Math 502: Abstract Algebra
Homework 9

Lawrence Tyler Rush
<me@tylerlogic.com>

January 5, 2014
http://coursework.tylerlogic.com/courses/upenn/math502/homework09

1

(a)

(b) Extra Credit

2

(a) Via Zorn's Lemma, prove that a nonzero $v \in V$ must be contained in some basis of V

Fix some nonzero $v \in V$ and define $\mathcal{L}$ to be the family of subsets of V defined by

$$\mathcal{L} = \{S \subseteq V \mid S \text{ is linearly independent and } v \in S\}$$

Then $\mathcal{L}$ is a poset regarding what it means to be a subset. Let $C = \{S_\alpha\}$ be a chain in $\mathcal{L}$. Because each S_α is linearly independent, then C must have a maximal element in $\mathcal{L}$ since V is a vector space and a basis in a vector space is a maximal linearly independent set. Thus by Zorn's Lemma, $\mathcal{L}$ must also have a maximal element. Such an element, by definition, is a basis of V.

(b)

For later contradiction, assume that j is not injective. Then there exist two distinct $v_1, v_2 \in V$ with $j(v_1) = j(v_2)$, or in other words $j(v_1)(\lambda) = j(v_2)(\lambda)$ for all $\lambda \in V^\vee$. This implies that

$$\lambda(v_1) = \lambda(v_2) \tag{2.1}$$

for all $\lambda \in V^\vee$.

However, let's let $\mathscr{B}$ be a basis containing v_1 but not containing v_2, and define $\gamma \in V^\vee$ to be the map

$$v \mapsto a$$

where a is the coordinate for v_1 when v is written in the basis $\mathscr{B}$. With this we have that $\gamma(v_1) = 1$ and $\gamma(v_2) = a$ with $a \neq 1$ since v_1 and v_2 were assumed distinct. Hence $\gamma(v_1) \neq \gamma(v_2)$, which contradicts equation 2.1. Therefore j must be injective.

***j* is *F*-linear** Let $v, u \in V$, $a, b \in F$ and $\lambda \in V^\vee$. Then we have the following

$$j(av_1 + bv_2)(\lambda) = \lambda(av_1 + bv_2) = a\lambda(v_1) + b\lambda(v_2) = aj(v_1)(\lambda) + bj(v_2)(\lambda) = \big(aj(v_1) + bj(v_2)\big)(\lambda)$$

by the linearity of λ. Hence j is F-linear.

(c)

(d)

3

(a) Show that (x, y) in $\mathbb{C}[x, y]$ is not principle.

For later contradiction, assume that (x, y) is principle. Then there is some element of $f \in C[x, y]$ that generates the ideal (x, y). Since $x \in (x, y)$ and $y \in (x, y)$, then f must divide both x and y. However, this is a contradiction with the fact that there is no element of $C[x, y]$ that divides both x and y.

(b) Extra Credit

(c) Extra Credit

4

Let $T \in \mathrm{End}_F(V)$ for a finite dimensional vector space V over a field F. Denote the dimension of V by n.

(a) Show that if T is diagonalizable, then T is semisimple.

Assume that T is diagonalizable. Then it's charateristic polynomial is

$$\mathrm{char}(T) = (\lambda - a_1)(\lambda - a_2) \cdots (\lambda - a_n) \tag{4.2}$$

where $a_1, \ldots, a_n$ are the diagonal entries of T as represented in the basis of its eigenvectors. Because the minimal polynomial divides the characteristic polynomial, according to the Caley-Hamilton theorem, then equation 4.2 indicates that the minimal polynomial is

$$(\lambda - b_1)(\lambda - b_2) \cdots (\lambda - b_m)$$

where $m \leq n$ and $b_1, \ldots, b_m$ are the distinct elements of $\{a_1, \ldots, a_n\}$. Hence T is semisimple.

(b)

(c)

(d) Extra Credit

(e) Extra Credit

5

Let R be a commuitative ring.

(a) Show that $R[x]$ is an integral domain iff R is an integral domain.

Let $R[x]$ be an integral domain. Let $a, b \in R$ be elements with $ab = 0$. Then a and b are also elements of $R[x]$ as constant polynomials. Hence either a or b must be zero as $R[x]$ has no zero divisors.

Conversely assume that R is an integral domain. ????

(b)

(c)

(d)

(e) Extra Credit

(f) Extra Credit

(g) Extra Credit
