Math 503: Abstract Algebra
Homework 2

Lawrence Tyler Rush
`<me@tylerlogic.com>`

In Collaboration With

Caitlin Beecham
Adam Freilich
Keaton Naff
Matt Weaver

February 6, 2014
http://coursework.tylerlogic.com/courses/upenn/math503/homework02

1

Let R be a commutative ring and G be a finite group.

(a) Show that $R \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ has a structure as a ring

The tensor product $R \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ is already an abelian group, so we need to find a multiplication operation $\cdot$ and show that $(R \otimes_{\mathbb{Z}} \mathbb{Z}[G], \cdot, 1_R \otimes 1_{\mathbb{Z}[G]})$ is a monoid as well as the distributive law holds.

Constructing Multiplication We first recognize that multiplication in a ring R' is some associative bilinear operation from $R' \times R' \to R'$. It's bilinear because of the distributive law, and associative so that the demands of the aforementioned monoid are met.

Thus because R and $\mathbb{Z}[G]$ are both rings, there exist such associative, bilinear maps $m_R : R \times R \to R$ and $m_{\mathbb{Z}[G]} : \mathbb{Z}[G] \times \mathbb{Z}[G] \to \mathbb{Z}[G]$. But then the universal property of tensor products yields $\overline{m_R} : R \otimes_{\mathbb{Z}} R \to R$ and $\overline{m_{\mathbb{Z}[G]}} : \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G] \to \mathbb{Z}[G]$ through which m_R and $m_{\mathbb{Z}[G]}$ factor, respectively. Thus we can combine these two to form the linear map $\overline{m_R} \otimes \overline{m_{\mathbb{Z}[G]}} : (R \otimes_{\mathbb{Z}} R) \otimes_{\mathbb{Z}} (\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G]) \to R \otimes_{\mathbb{Z}} \mathbb{Z}[G]$. Because of the commutativity and associativity of the tensor product, there exists an isomorphism $\alpha : (R \otimes_{\mathbb{Z}} \mathbb{Z}[G]) \otimes (R \otimes_{\mathbb{Z}} \mathbb{Z}[G]) \to (R \otimes_{\mathbb{Z}} R) \otimes_{\mathbb{Z}} (\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G])$ such that $\alpha((r \otimes x) \otimes (s \otimes y)) = (r \otimes s) \otimes (x \otimes y)$ for all $r, s \in R$ and $x, y \in \mathbb{Z}[G]$. Now the composition $\overline{m_R} \otimes \overline{m_{\mathbb{Z}[G]}} \circ \alpha : (R \otimes_{\mathbb{Z}} \mathbb{Z}[G]) \otimes (R \otimes_{\mathbb{Z}} \mathbb{Z}[G]) \to R \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ is a linear map, which, via the universal property of tensor products, yields a bilinear map $m : (R \otimes_{\mathbb{Z}} \mathbb{Z}[G]) \times (R \otimes_{\mathbb{Z}} \mathbb{Z}[G]) \to R \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ through which $\overline{m_R} \otimes \overline{m_{\mathbb{Z}[G]}} \circ \alpha$ factors. The map m is our desired multiplication.

Associativity of Multiplication Because m is bilinear, it satisfies the distributive laws and thus we need only show that m is associative. Also because m is bilinear, for arbitrary $\sum_i r_i \otimes x_i, \sum_j s_j \otimes y_j \in R \otimes_{\mathbb{Z}} \mathbb{Z}[G]$

$$m\left(\sum_i r_i \otimes x_i\ ,\ \sum_j s_j \otimes y_j\right) = \sum_i \sum_j m\left(r_i \otimes x_i, s_j \otimes y_j\right)$$

which implies that we need only determine that $m(r \otimes x, m(s \otimes y, t \otimes z)) = m(m(r \otimes x, s \otimes y), t \otimes z)$ for all $r, s, t \in R$ and $x, y, z \in \mathbb{Z}[G]$. So finally, through the use of the associativity of m_R and $m_{\mathbb{Z}[G]}$ the following sequence of equations shows m to be associative. For clarity we set $\varphi = \overline{m_R} \otimes \overline{m_{\mathbb{Z}[G]}}$

$$\begin{aligned}
m\Big(r \otimes x, m\big(s \otimes y, t \otimes z\big)\Big) &= \varphi \circ \alpha\Big(\big(r \otimes x\big) \otimes \varphi \circ \alpha\big((s \otimes y) \otimes (t \otimes z)\big)\Big) \\
&= \varphi \circ \alpha\Big(\big(r \otimes x\big) \otimes \varphi\big((s \otimes t) \otimes (y \otimes z)\big)\Big) \\
&= \varphi \circ \alpha\Big(\big(r \otimes x\big) \otimes \big(\overline{m_R}(s \otimes t) \otimes \overline{m_{\mathbb{Z}[G]}}(y \otimes z)\big)\Big) \\
&= \varphi \circ \alpha\Big(\big(r \otimes x\big) \otimes \big(m_R(s,t) \otimes m_{\mathbb{Z}[G]}(y,z)\big)\Big) \\
&= \varphi\Big(\big(r \otimes m_R(s,t)\big) \otimes \big(x \otimes m_{\mathbb{Z}[G]}(y,z)\big)\Big) \\
&= \overline{m_R}\big(r \otimes m_R(s,t)\big) \otimes \overline{m_{\mathbb{Z}[G]}}\big(x \otimes m_{\mathbb{Z}[G]}(y,z)\big) \\
&= m_R(r, m_R(s,t)) \otimes m_{\mathbb{Z}[G]}(x, m_{\mathbb{Z}[G]}(y,z)) \\
&= m_R(m_R(r,s),t) \otimes m_{\mathbb{Z}[G]}(m_{\mathbb{Z}[G]}(x,y),z) \\
&= \overline{m_R}(m_R(r,s) \otimes t) \otimes \overline{m_{\mathbb{Z}[G]}}(m_{\mathbb{Z}[G]}(x,y) \otimes z) \\
&= \varphi\Big(\big(m_R(r,s) \otimes t\big) \otimes \big(m_{\mathbb{Z}[G]}(x,y) \otimes z\big)\Big) \\
&= \varphi \circ \alpha\Big(\big(m_R(r,s) \otimes m_{\mathbb{Z}[G]}(x,y)\big) \otimes \big(t \otimes z\big)\Big) \\
&= \varphi \circ \alpha\Big(\big(\overline{m_R}(r \otimes s) \otimes \overline{m_{\mathbb{Z}[G]}}(x \otimes y)\big) \otimes \big(t \otimes z\big)\Big) \\
&= \varphi \circ \alpha\Big(\varphi\big((r \otimes s) \otimes (x \otimes y)\big) \otimes \big(t \otimes z\big)\Big) \\
&= \varphi \circ \alpha\Big(\varphi \circ \alpha\big((r \otimes x) \otimes (s \otimes y)\big) \otimes \big(t \otimes z\big)\Big) \\
&= \varphi \circ \alpha\Big(m\big(r \otimes x, s \otimes y\big) \otimes \big(t \otimes z\big)\Big) \\
&= m\Big(m\big(r \otimes x, s \otimes y\big), t \otimes z\Big)
\end{aligned}$$

Thus $R \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ has a ring structure.

(b) Are $R \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ and $R[G]$ isomorphic?

Because R is a free R-module of rank one, $\mathbb{Z}[G]$ is a free $\mathbb{Z}$-module of rank $\#G$, and $R[G]$ is a free R-module of rank $\#G$, then we know immediately that $R \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ and $R[G]$ are isomorphic as R-modules, since their ranks are the same. Furthermore, any homomorphism which takes basis elements to distinct basis elements will be an R-linear isomorphism. Thus if we can find such an R-module isomorphism and go on to show that it preserves multiplication between elements of the domain and codomain, then it will also be a ring isomorphism. We endeavor to find such an isomorphism.

So define $\alpha : R \times \mathbb{Z}[G] \to R[G]$ to be the map $(r, x) \mapsto rx$. Through use of the properties of $R[G]$, for all $r, r_1, r_2 \in R$, $x, x_1, x_2 \in R[G]$, and $n \in \mathbb{Z}$

$$\alpha(r_1 + r_2, x) = (r_1 + r_2)x = r_1x + r_2x = \alpha(r_1, x) + \alpha(r_2, x)$$
$$\alpha(r, x_1 + x_2) = r(x_1 + x_2) = rx_1 + rx_2 = \alpha(r, x_1) + \alpha(r, x_2)$$
$$\alpha(nr, x) = (nr)x = \text{signum}(n) \underbrace{(r + \cdots + r)}_{|n| \text{ times}} x = \text{signum}(n) \underbrace{rx + \cdots + rx}_{|n| \text{ times}} = \text{signum}(n) r \underbrace{(x + \cdots + x)}_{|n| \text{ times}} = r(nx) = \alpha(r, nx)$$

by which α is $\mathbb{Z}$-bilinear. Therefore the universal properties of tensor products yields $\overline{\alpha} : R \otimes \mathbb{Z}[G] \to R[G]$ such that $\alpha = \overline{\alpha} \circ i$ where i is the inclusion map. Moreover, $\overline{\alpha}$ is an isomorphism of modules due to it's mapping basis elements to distinct basis elements: $\overline{\alpha}(1 \otimes [g]) = \alpha(1, [g]) = [g]$ for each $g \in G$. It remains to be shown that $\overline{\alpha}$ preserves the operation of multiplication. The following yields that fact for arbitrary elements $\sum_i r_i \otimes x_i$ and $\sum_j s_j \otimes y_j$ in $R \otimes \mathbb{Z}[G]$

$$\begin{aligned}
\overline{\alpha}\left(\left(\sum_i r_i \otimes x_i\right)\left(\sum_j s_j \otimes y_j\right)\right) &= \overline{\alpha}\left(\sum_i \sum_j (r_i \otimes x_i)(s_j \otimes y_j)\right) \\
&= \overline{\alpha}\left(\sum_i \sum_j (r_i s_j \otimes x_i y_j)\right) \\
&= \sum_i \sum_j \overline{\alpha}(r_i s_j \otimes x_i y_j) \\
&= \sum_i \sum_j \alpha(r_i s_j, x_i y_j) \\
&= \sum_i \sum_j r_i s_j x_i y_j \\
&= \left(\sum_i r_i x_i\right)\left(\sum_j s_j y_j\right) \\
&= \left(\sum_i \alpha(r_i, x_i)\right)\left(\sum_j \alpha(s_j, y_j)\right) \\
&= \left(\sum_i \overline{\alpha}(r_i \otimes x_i)\right)\left(\sum_j \overline{\alpha}(s_j \otimes y_j)\right) \\
&= \overline{\alpha}\left(\sum_i r_i \otimes x_i\right)\overline{\alpha}\left(\sum_j s_j \otimes y_j\right)
\end{aligned}$$

Hence, $R \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ and $R[G]$ are isomorphic rings.

2

Let M and N be two left R-modules over a non-commutative ring R. Define $M \odot_R N$ to be the quotient of $M \otimes_{\mathbb{Z}} N$ by it's submodule which is generated by all elements of the form $(r \cdot m) \otimes n - m \otimes (r \cdot n)$ where $m \in M$, $n \in N$, and $r \in R$. We will refer to this submodule as S.

(a)

Define $\alpha : M \times N \to M \odot_R N$ to be the compositions of the canonical map $i_1 : M \times N \to M \otimes_{\mathbb{Z}} N$ and the quotient map $i_2 : M \otimes_{\mathbb{Z}} N \to M \odot_R N$.

Let Q be an R-module and $\gamma : M \times N \to Q$ be a R-bilinear map. Therefore Q is an abelian group and γ is also a $\mathbb{Z}$-bilinear map. Thus the universal property of tensor products gives us the existence of a unique $\gamma' \in \mathrm{Hom}_{\mathrm{grp}}(M \otimes_{\mathbb{Z}} N, Q)$ such that $\gamma = \gamma' \circ i_1$.

Now since S is generated by elements of the form $(r \cdot m) \otimes n - m \otimes (r \cdot n)$ where $m \in M$, $n \in N$, and $r \in R$ and

$$\begin{aligned}
\gamma'((r \cdot m) \otimes n - m \otimes (r \cdot n)) &= \gamma'((r \cdot m) \otimes n) - \gamma'(m \otimes (r \cdot n)) \\
&= \gamma(r \cdot m, n) - \gamma(m, r \cdot n) \\
&= r\gamma(m, n) - r\gamma(m, n) \\
&= 0
\end{aligned}$$

then $S \leq \ker \gamma'$, by which the universal property of quotient modules yields a unique map $\beta \in \mathrm{Hom}_R(M \odot_R N, Q)$ such that $\gamma' = \beta \circ i_2$. Hence $\gamma = \gamma' \circ i_2 = \beta \circ i_2 \circ i_1 = \beta \circ \alpha$.

Finally, the existence and uniqueness of both γ' and β, demand that the map $\beta \mapsto \beta \circ \alpha$ is bijective.

(b) What is $M \odot_R N$ when M, N are R-modules of rank one

Let M and N each be left R-modules of rank 1 with generators m and n, respectively. Then for the R-module $M \odot_R N$ and for an arbitrary element $\sum_i m_i \odot n_i \in M \odot_R N$

$$\begin{aligned} \sum_i m_i \odot n_i &= \sum_i r_i m \odot s_i n \\ &= \sum_i s_i r_i m \odot n \\ &= \left(\sum_i s_i r_i \right) (m \odot n) \end{aligned}$$

while similarly

$$\begin{aligned} \sum_i m_i \odot n_i &= \sum_i r_i m \odot s_i n \\ &= \sum_i m \odot r_i s_i n \\ &= \sum_i ((r_i s_i) m \odot n) \\ &= \left(\sum_i r_i s_i \right) (m \odot n) \end{aligned}$$

Either one of these implies that $M \odot_R N$ is isomorphic as a R-module to a subring of R, not necessarily proper. However, combining the two results brings

$$\left(\sum_i s_i r_i \right) (m \odot n) = \left(\sum_i r_i s_i \right) (m \odot n)$$

to light, from which we deduce that

$$M \odot_R N \cong {}^{R}/_{S}$$

where S is the subring of R generated by elements of the form $rs - sr$. Note that since R is non-commutative, then S will not simply be zero.

(c) Give an R such that $M \odot_R N$ is zero for M, N in (b)

Begin with a non-commutative ring, say $M_2(\mathbb{R})$, and let $M = N = \mathbb{R}$. Set R to be the ring generated by elements of the form $AB - BA$ for $A, B \in M_2(\mathbb{R})$. Then certainly, given the previous part of this problem, the R-module $M \odot_R N$ will be zero because it will be isomorphic to ${}^{R}/_{R}$ in this case.

3

Let G be a finite group with subgroup H. Let F be a field.

(a)

Because F is a field, then the group rings $F[G \times G]$ and $F[G]$ are vector spaces over F with dimension $(\#G)^2$ and $\#G$, respectively. Hence $F[G] \otimes_F F[G]$ is also a vector space of dimension $(\#G)^2$. So $F[G \times G]$ and $F[G] \otimes_F F[G]$ are isomorphic. Furthermore, because $\{[(x,y)] | x, y \in G\}$ is a basis for $F[G \times G]$ and $\{[x] \otimes [y] | x, y \in G\}$ is a basis for $F[G] \otimes_F F[G]$, then the map from $F[G \times G]$ to $F[G] \otimes_F F[G]$ defined by $[(x,y)] \mapsto [x] \otimes [y]$ is an isomorphism, and uniquely so.

(b)

Again, because F is a field, $F[G]$ and $F[G] \otimes_F F[G]$ are vector spaces over F. Furthermore, because $\{[x] | x \in G\}$ and $\{[x] \otimes [y] | x, y \in G\}$ are bases for $F[G]$ and $F[G] \otimes_F F[G]$, respectively, then defining $\alpha : F[G] \to F[G] \otimes_F F[G]$ by $[x] \mapsto [x] \otimes [x]$ makes α the unique injective *linear* homomorphism from $F[G]$ to $F[G] \otimes_F F[G]$. In order for α to be an F-algebra homomorphism, it remains only to prove that $\alpha([x][y]) = \alpha([x])\alpha([y])$ for all $[x], [y] \in F[G]$. The following yields that property:

$$\begin{aligned} \alpha([x][y]) &= \alpha([xy]) \\ &= [xy] \otimes [xy] \\ &= [x][y] \otimes [x][y] \\ &= ([x] \otimes [x])([y] \otimes [y]) \\ &= \alpha([x])\alpha([y]) \end{aligned}$$

(c)

Since V and W are left $F[G]$-modules, then they a free modules each of rank $\#G$. Since the tensor product of free modules is also a free module, then $V \otimes_F W$ is also a left $R[G]$-module, however, it has rank $(\#G)^2$.

Fix an $x \in G$. Given that $\rho_V(x) \in GL(V)$ and $\rho_W(x) \in GL(W)$ then we can define $\rho_{V \otimes W} : G \to GL(V)$ by $\rho_{V \otimes W}(x) = \rho_V(x) \otimes \rho_W(x)$

So set A to be the matrix representation of $\rho_V(x)$ in the basis $\{[g] | g \in G\}$. Similarly, set B to be the matrix representation of $\rho_W(x)$ in the same basis. Also set C to be the matrix representation of $\rho_{V \otimes W}(x)$ in the basis $\{[g] \otimes 1 | g \in G\} \cup \{1 \otimes [g] | g \in G\}$. Then

$$C = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & & \\ \vdots & & \ddots & \\ a_{n1}B & a_{n2}B & & a_{nn}B \end{pmatrix}$$

where $n = \#G$ and $(a_{ij}) = A$. Therefore we finally arrive at

$$\begin{aligned}
\operatorname{Tr}_F(\rho_{V\otimes W}(x)) &= \operatorname{Tr}_F(C) \\
&= \operatorname{Tr}_F\left(\sum_{i=1}^{n} a_{ii}B\right) \\
&= \sum_{i=1}^{n} a_{ii}\operatorname{Tr}_F(B) \\
&= \operatorname{Tr}_F(B)\sum_{i=1}^{n} a_{ii} \\
&= \operatorname{Tr}_F(B)\operatorname{Tr}_F(A) \\
&= \operatorname{Tr}_F(A)\operatorname{Tr}_F(B) \\
&= \operatorname{Tr}_F(\rho_V(x))\operatorname{Tr}_F(\rho_W(x))
\end{aligned}$$

(d) Extra Credit

4

(a) Show that there is a natural ring isomorphism between $\mathbb{Z}[x, y]$ and $\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[x]$

The rings $\mathbb{Z}[x, y]$ and $\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[x]$ are free $\mathbb{Z}$-modules with bases $\{x^n | n \in \mathbb{Z}_{\geq 0}\} \cup \{y^n | n \in \mathbb{Z}_{\geq 0}\}$ and $\{x^n \otimes 1 | n \in \mathbb{Z}_{\geq 0}\} \cup \{1 \otimes x^n | n \in \mathbb{Z}_{\geq 0}\}$, respectively. These two bases have the same cardinality, so any homomorphism that maps the elements of one basis to distinct elements of the other will be a $\mathbb{Z}$-module isomorphism. So once we find such a module isomorphism, we need only show that it preserves the multiplication operation in order to obtain a ring isomorphism.

Define $\alpha : \mathbb{Z}[x] \times \mathbb{Z}[x] \to \mathbb{Z}[x, y]$ by $\big(f(x), g(x)\big) \mapsto f(x)g(y)$. Then for all $f, f_1, f_2, g, g_1, g_2 \in \mathbb{Z}[x]$ and $n \in \mathbb{Z}$ we have the following through heavy use of ring properties of $\mathbb{Z}[x, y]$.

$$\begin{aligned}
\alpha(f_1(x) + f_2(x), g(x)) &= (f_1(x) + f_2(x))g(y) \\
&= f_1(x)g(y) + f_2(x)g(y) \\
&= \alpha(f_1(x), g(x)) + \alpha(f_2(x), g(x)) \\
\alpha(f(x), g_1(x) + g_2(x)) &= f(x)(g_1(y) + g_2(y)) \\
&= f(x)g_1(y) + f(x)g_2(y) \\
&= \alpha(f(x)g_1(y)) + \alpha(f(x)g_2(y)) \\
\alpha(nf(x), g(x)) &= \operatorname{signum}(n)\underbrace{(f(x) + \cdots + f(x))}_{|n| \text{ times}}g(y) \\
&= \operatorname{signum}(n)\underbrace{f(x)g(y) + \cdots + f(x)g(y)}_{|n| \text{ times}} \\
&= \operatorname{signum}(n)f(x)\underbrace{(g(y) + \cdots + g(y))}_{|n| \text{ times}} \\
&= \alpha(f(x), ng(x))
\end{aligned}$$

which implies that α is $\mathbb{Z}$-bilinear. Thus the universal property of tensor products gives us $\overline{\alpha} : \mathbb{Z}[x] \otimes \mathbb{Z}[x] \to \mathbb{Z}[x, y]$ such that $\alpha = \overline{\alpha} \circ i$ where i is the inclusion map. Now $\overline{\alpha}(x^n \otimes 1) = \alpha(x^n, 1) = x^n$ and $\overline{\alpha}(1 \otimes x^n) = \alpha(1, x^n) = y^n$, making $\overline{\alpha}$ a $\mathbb{Z}$-module isomorphism. We now only to multiplication to be preserved by $\overline{\alpha}$. This is shown by the following for arbitrary elements $\sum_i f_i \otimes g_i,\ \ \sum_j f_j \otimes g_j \in \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[x]$

$$\begin{aligned}
\overline{\alpha}\left(\left(\sum_i f_i \otimes g_i\right)\left(\sum_j f_j \otimes g_j\right)\right) &= \overline{\alpha}\left(\sum_i \sum_j (f_i \otimes g_i)(f_j \otimes g_j)\right) \\
&= \overline{\alpha}\left(\sum_i \sum_j (f_i f_j \otimes g_i g_j)\right) \\
&= \sum_i \sum_j \overline{\alpha}(f_i f_j \otimes g_i g_j) \\
&= \sum_i \sum_j \alpha(f_i f_j, g_i g_j) \\
&= \sum_i \sum_j f_i(x) f_j(x) g_i(y) g_j(y) \\
&= \left(\sum_i f_i(x) g_i(y)\right)\left(\sum_j f_j(x) g_j(y)\right) \\
&= \left(\sum_i \alpha(f_i, g_i)\right)\left(\sum_j \alpha(f_j, g_j)\right) \\
&= \left(\sum_i \overline{\alpha}(f_i \otimes g_i)\right)\left(\sum_j \overline{\alpha}(f_j \otimes g_j)\right) \\
&= \overline{\alpha}\left(\sum_i f_i \otimes g_i\right)\overline{\alpha}\left(\sum_j f_j \otimes g_j\right)
\end{aligned}$$

Hence $\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[x]$ is naturally isomorphic to $\mathbb{Z}[x, y]$.

(b)

Let $c : \mathbb{Z}[x] \to \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[x]$ be a ring homomorphism such that $c(x) = x \otimes 1 + 1 \otimes x$ for the polynomial $x \in \mathbb{Z}[x]$. Therefore, for $n \in \mathbb{Z}$, $c(x^n) = (c(x))^n$. Since $\{1, x, x^2, \ldots\}$ is a basis for $\mathbb{Z}[x]$, then $c(x) = x \otimes 1 + 1 \otimes x$ completely defines c. Hence, c is unique.

(c)

First note that for $n \in \mathbb{Z}$ we have a formula for $c(x^n)$

$$c(x^n) = (x \otimes 1 + 1 \otimes x)^n = \sum_{i=0}^{n} \binom{n}{i} (x \otimes 1)^{n-i} (1 \otimes x)^i = \sum_{i=0}^{n} \binom{n}{i} (x^{n-i} \otimes 1)(1 \otimes x^i) = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} \otimes x^i$$

Note that the last line could also be written as $\sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i}$. We will make use of both. Then, for any $f(x) = \in \mathbb{Z}[x]$, $c(f(x)) = f(x \otimes 1 + 1 \otimes x) = \sum_n a_n \sum_{i=0}^{n} \binom{n}{i} x^{n-i} \otimes x^i$ when $f(x) = \sum_n a_n x^n$. Given these results, we obtain the following

$$
\begin{aligned}
\alpha \circ (1 \otimes c) \circ c(f) &= \alpha(1 \otimes c(f(x \otimes 1 + 1 \otimes x))) \\
&= \alpha\left(1 \otimes c\left(\sum_n a_n \sum_{i=0}^{n} \binom{n}{i} x^{n-i} \otimes x^i\right)\right) \\
&= \alpha\left(1 \otimes c\left(\sum_n \sum_{i=0}^{n} \left(a_n \binom{n}{i} x^{n-i}\right) \otimes x^i\right)\right) \\
&= \alpha\left(\sum_n \sum_{i=0}^{n} \left(a_n \binom{n}{i} x^{n-i}\right) \otimes c\left(x^i\right)\right) \\
&= \alpha\left(\sum_n \sum_{i=0}^{n} \left(a_n \binom{n}{i} x^{n-i}\right) \otimes \left(\sum_{k=0}^{i} \binom{i}{k} x^{i-k} \otimes x^k\right)\right) \\
&= \alpha\left(\sum_n \sum_{i=0}^{n} \sum_{k=0}^{i} a_n \binom{n}{i} \binom{i}{k} \left(x^{n-i} \otimes \left(x^{i-k} \otimes x^k\right)\right)\right) \\
&= \sum_n \sum_{i=0}^{n} \sum_{k=0}^{i} a_n \binom{n}{i} \binom{i}{k} \left(\left(x^{n-i} \otimes x^{i-k}\right) \otimes x^k\right)
\end{aligned}
$$

???? Seems like there should be a way to manipulate the coefficients above so that $(c \otimes 1) \circ c$ results, but I can't figure out how.

(d)

(e)

(f)

(g) Extra Credit

(h) Extra Credit
