Math 508: Advanced Analysis
Homework 1

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September 10, 2014
http://coursework.tylerlogic.com/courses/upenn/math508/homework01

1 For rational $r \neq 0$ and irrational x, prove that $r + x$ and rx are irrational.

Since r is rational, both $1/r$ and $-r$ are rational. Thus, if $r + x$ were rational then

$$x = (r + x) + (-r)$$

would imply x is rational by the closure of addition in the field of rationals; but since x is assumed to be irrational, so must be $r + x$.

Similarly, rx must be irrational since

$$x = (rx)(1/r)$$

and the field of rationals is closed under multiplication.

2 Prove that there's no rational number with a square of 12

Lemma 2.1. *The square root of 3 is irrational.*

Proof. Assume for later contradiction that 3 is rational. Then we can find integers m, n which are not both even such that $m/n = \sqrt{3}$. Therefore

$$m^2 = 3n^2$$

This equation then demands that if n^2 is even, then so must be m^2 (and therefore m), and similarly if n^2 is odd, then so must be m^2 (and therefore m). Because m and n were assumed to not both be even, then m and n must both be odd. Thus we can set $m = 2p + 1$ and $n = 2q + 1$. Furthermore the above equation implies that

$$\begin{aligned}
(2p+1)^2 &= 3(2q+1)^2 \\
4p^2 + 4p + 1 &= 12q^2 + 12q + 3 \\
4p^2 + 4p &= 12q^2 + 12q + 2 \\
2p^2 + 2p &= 6q^2 + 6q + 1 \\
2(p^2 + p) &= 2(3q^2 + 3q) + 1
\end{aligned}$$

however, the left-hand side of the last line above is even and the right-hand side is odd; a contradiction. So 3 must be irrational. □

Since $\sqrt{12} = 2\sqrt{3}$ the previous problem and the above lemma together imply that $\sqrt{12}$ is irrational.

3 Prove the following from the field axioms for multiplication.

(a) If $x \neq 0$ and $xy = xz$, then $y = z$

Let $x \neq 0$ and $xy = xz$, then

$$y = (1)y = \left(\frac{1}{x}x\right) y = \frac{1}{x}(xy) = \frac{1}{x}(xz) = \left(\frac{1}{x}x\right) z = 1(z) = z$$

(b) If $x \neq 0$ and $xy = x$, then $y = 1$

This follows from the previous part for $z = 1$.

(c) If $x \neq 0$ and $xy = 1$, then $y = 1/x$

Again, this follows from the first part of this problem for $z = 1/x$.

(d) If $x \neq 0$ then $1/(1/x) = x$

Let $x \neq 0$, then

$$x = (1)x = \left(\frac{1}{1/x}(1/x)\right)x = \frac{1}{1/x}\left((1/x)x\right) = \frac{1}{1/x}(1) = \frac{1}{1/x}$$

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For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with the Euclidean distance, we have

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 &= (\mathbf{x}\cdot\mathbf{x} + 2\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y}) + (\mathbf{x}\cdot\mathbf{x} - 2\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y}) \\ &= \mathbf{x}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} + \mathbf{x}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} \\ &= 2\mathbf{x}\cdot\mathbf{x} + 2\mathbf{y}\cdot\mathbf{y} \\ &= 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2 \end{aligned}$$

Geometrically, if $|\mathbf{x}|$ and $|\mathbf{y}|$ are the two lengths of two adjacent sides of a parallelogram, then $|\mathbf{x}+\mathbf{y}|$ and $|\mathbf{x}-\mathbf{y}|$ would be the lengths of the two diagonals. So this equation reveals that the sum of the squares of the diagonals of a parallelogram is equal to twice the sum of the squares of its sides.

5 Fix $a, b \in \mathbb{R}^k$. Show there exist $c \in \mathbb{R}^k$ and $r \in \mathbb{R}$ such that $\forall x \in \mathbb{R}^k$, $|x-a| = \lambda|x-b|$ iff $|x-c| = r$, for $\lambda > 1$

Letting $(\lambda^2-1)c = \lambda^2 b - a$ and $r = \lambda(\lambda^2-1)^{-1}|b-a|$ we obtain

$$(\lambda^2-1)^2|c|^2 = |\lambda^2 b - a|^2 = \lambda^4|b|^2 - 2\lambda^2\langle b, a\rangle + |a|^2 \tag{5.1}$$

Now assuming that $|x-a| = \lambda|x-b|$ for some point $x \in \mathbb{R}^k$ we get the following sequence of equations.

$$\begin{aligned} \lambda^2|x-b|^2 &= |x-a|^2 \\ \lambda^2\left(|x|^2 - 2\langle x, b\rangle + |b|^2\right) &= |x|^2 - 2\langle x, a\rangle + |a|^2 \\ (\lambda^2-1)|x|^2 - 2\lambda^2\langle x, b\rangle + 2\langle x, a\rangle &= |a|^2 - \lambda^2|b|^2 \\ (\lambda^2-1)|x|^2 - 2\left\langle x, \lambda^2 b + a\right\rangle &= |a|^2 - \lambda^2|b|^2 \\ (\lambda^2-1)|x|^2 - 2\left\langle x, (\lambda^2-1)c\right\rangle &= |a|^2 - \lambda^2|b|^2 \\ (\lambda^2-1)|x|^2 - 2\left\langle x, (\lambda^2-1)c\right\rangle + (\lambda^2-1)|c|^2 &= (\lambda^2-1)|c|^2 + |a|^2 - \lambda^2|b|^2 \\ (\lambda^2-1)\left(|x|^2 - 2\langle x, c\rangle + |c|^2\right) &= (\lambda^2-1)|c|^2 + |a|^2 - \lambda^2|b|^2 \\ (\lambda^2-1)|x-c|^2 &= (\lambda^2-1)|c|^2 + |a|^2 - \lambda^2|b|^2 \end{aligned}$$

At this point, Equation 5.1 yields the fact that $(\lambda^2-1)|c|^2 = (\lambda^2-1)^{-1}\left(\lambda^4|b|^2 - 2\lambda^2\langle b,a\rangle + |a|^2\right)$, so making that substitution into the last line from the sequence of equations above, we continue:

$$\begin{aligned}
(\lambda^2-1)|x-c|^2 &= (\lambda^2-1)^{-1}\left(\lambda^4|b|^2 - 2\lambda^2\langle b,a\rangle + |a|^2\right) + |a|^2 - \lambda^2|b|^2 \\
(\lambda^2-1)|x-c|^2 &= (\lambda^2-1)^{-1}\left((\lambda^2-1)^2|c|^2 + (\lambda^2-1)|a|^2 - (\lambda^2-1)\lambda^2|b|^2\right) \\
|x-c|^2 &= (\lambda^2-1)^{-2}\left((\lambda^2-1)^2|c|^2 + (\lambda^2-1)|a|^2 - (\lambda^2-1)\lambda^2|b|^2\right) \\
|x-c|^2 &= (\lambda^2-1)^{-2}\left(\left(\lambda^4|b|^2 - 2\lambda^2\langle b,a\rangle + |a|^2\right) + (\lambda^2-1)|a|^2 - (\lambda^2-1)\lambda^2|b|^2\right) \\
|x-c|^2 &= (\lambda^2-1)^{-2}\left(\lambda^2|b|^2 - 2\lambda^2\langle b,a\rangle + \lambda^2|a|^2\right) \\
|x-c|^2 &= \lambda^2(\lambda^2-1)^{-2}\left(|b|^2 - 2\langle b,a\rangle + |a|^2\right) \\
|x-c|^2 &= \lambda^2(\lambda^2-1)^{-2}|b-a|^2 \\
|x-c|^2 &= \left(\lambda(\lambda^2-1)^{-1}|b-a|\right)^2 \\
|x-c|^2 &= r^2
\end{aligned}$$

With the final line above we must have $|x-c| = r$ since both $|x-c|$ and r are positive values (r being positive because $\lambda > 1$). So we have that $|x-a| = \lambda|x-b|$ implies that $|x-c| = r$; however, reversing the order of equations above also reveals that $|x-c| = r$ implies $|x-a| = \lambda|x-b|$, giving us the "iff" we desire.

What happens when $\boldsymbol{\lambda = 1}$? When $\lambda = 1$, we'd be interested in all points $x \in \mathbb{R}^k$ where $|x-a| = |x-b|$. However the points that satisfy this are the points in the hyper-plane that perpendicularly bisects the line segment, which of course has no notion of a radius since it is a hyper-plane and not a hyper-sphere. Hence it makes no sense to inquire about a sphere of radius $r = \lambda(\lambda^2-1)^{-1}|b-a|$; this is dually supported by the denominater, (λ^2-1), having a value of zero when $\lambda = 1$. In a silly way of thinking, it's as if the sphere gets so big (as λ approaches 1) that it "becomes a plane".

6 Prove that a union of a countable number of finite sets is countable

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Lemma 6.1. *If $S_1, S_2, S_3, \ldots$ are finite sets that are mutually disjoint, then $\cup_{j=1}^{\infty} S_i$ is countable.*

Proof. First note that if any S_i is the empty set it can simply be neglected. So fix an ordering of each of the elements of each set S_i and label them as follows

$$\begin{aligned}
S_1 &= \{x_{11}, x_{12}, \ldots, x_{1n_1}\} \\
S_2 &= \{x_{21}, x_{22}, \ldots, x_{2n_2}\} \\
S_3 &= \{x_{31}, x_{32}, \ldots, x_{3n_3}\} \\
&\vdots
\end{aligned}$$

where $n_i = |S_i|$. With this labeling, the following correspondence shows that $\cup_{j=1}^{\infty} S_j$ is countable

$$\begin{aligned}
x_{11} &\mapsto 1 \\
x_{12} &\mapsto 2 \\
&\vdots \\
x_{1n_1} &\mapsto n_1 \\
x_{21} &\mapsto n_1 + 1 \\
x_{22} &\mapsto n_1 + 2 \\
&\vdots \\
x_{2n_2} &\mapsto n_1 + n_2 \\
x_{31} &\mapsto n_1 + n_2 + 1 \\
&\vdots
\end{aligned}$$

$\square$

Let $S_1, S_2, S_3, \ldots$ each be a finite set. The problem here is that these sets are not mutually disjoint, but nevertheless, we can define the sets $T_1, T_2, T_3, \ldots$ as $T_1 = S_1$ and $T_n = S_n - \cap_{j=1}^{n-1} S_j$ and furthermore, $\cap_{j=1}^{n-1} T_j = \cap_{j=1}^{n-1} S_j$. So due to the above lemma, $\cap_{j=1}^{n-1} T_j$ is countable and therefore so is $A = \cap_{j=1}^{n-1} S_j$.

7 Prove that the algebraic numbers are countable

Define the sets $S_1, S_2, S_3, \ldots$ by

$$S_n = \left\{ z \in \mathbb{C} \;\middle|\; \exists\, a_0, \ldots, a_n \in \mathbb{Z} \cap [-n, n] \;\ni a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0 \right\}$$

Now with this definition, the set $A = \cup_{i=1}^{\infty} S_i$ will contain only algebraic numbers, but it will furthermore contain *all* of the algebraic numbers since if z is the root of $a_n x^n + \cdots + a_1 x + a_0$ and some a_i has $|a_i| > n$, then z will be in the S_N where $N = \max(|a_0|, \ldots, |a_n|)$ since z will also be a root of $0x^N + 0x^{N-1} + \cdots + 0x^{n+1} + a_n x^n + \cdots + a_1 x + a_0$. Also, because a polynomial of degree n has at most n roots and the set $\mathbb{Z} \cap [-n, n]$ is finite for any n, then each of the sets S_i are finite. Therefore, the previous problem gives us that A (the set of algebraic numbers) is a countable set.

8 Defining real exponents

Let $b > 1$ be real and $m, n, p, q \in \mathbb{Z}$, $r \in \mathbb{Q}$ such that $r = m/n = p/q$

(a) Prove $(b^m)^{1/n} = (b^p)^{1/q}$

Since $m/n = p/q$, then $mq = np$ and through use of the laws of integer exponents we obtain

$$(b^m)^{1/n} = \left(\left(\left(b^{1/q} \right)^q \right)^m \right)^{1/n} = \left(\left(b^{1/q} \right)^{mq} \right)^{1/n} = \left(\left(b^{1/q} \right)^{np} \right)^{1/n} = \left(\left((b^p)^{1/q} \right)^n \right)^{1/n} = (b^p)^{1/q}$$

So it makes sense to define $b^r = (b^m)^{1/n}$

(b) Prove $b^{r+s} = b^r b^s$ for rational r, s

Let $s = a/b$ for $a, b \in \mathbb{Z}$. Then we have

$$r + s = m/n + a/b = \frac{mb + an}{nb}$$

which in turn allows

$$\begin{aligned} b^{r+s} &= \left(b^{mb+an} \right)^{1/nb} \\ &= \left(b^{mb} b^{an} \right)^{1/nb} \\ &= \left(b^{mb} \right)^{1/nb} \left(b^{an} \right)^{1/nb} \\ &= \left(\left((b^m)^{1/n} \right)^b \right)^{1/b} \left(\left((b^a)^{1/b} \right)^n \right)^{1/n} \\ &= (b^m)^{1/n} (b^a)^{1/b} \\ &= b^r b^s \end{aligned}$$

through the use of properties of integer exponents and the previous part of this problem.

(c) Prove that $b^r = \sup B(r)$

Define $B(r) = \{b^t \mid t \in \mathbb{Q} \text{ and } t \le r\}$. Because $b^r \in B(r)$, $f(t) = b^t$ is monitonically increasing since $b > 0$, and any $b^t \in B(r)$ for rational t would have $t \le r$, then b^r must be $\sup B(r)$.

(d) Defining $b^x = \sup B(x)$ $\forall x \in \mathbb{R}$, show that $b^{x+y} = b^x b^y$ for $x, y \in \mathbb{R}$

Define $B(x, y) = \{b^s b^t \mid s, t \in \mathbb{Q} \text{ and } s \le x, t \le y\}$. By an argument identical to the previous part of this problem, we have $\sup B(x, y) = b^x b^y$. However $B(x, y)$ is also $\{b^{s+t} \mid s, t \in \mathbb{Q} \text{ and } s \le x, t \le y\}$, which is the same as $\{b^t \mid t \in \mathbb{Q} \text{ and } t \le x + y\}$. Of course this last set is just the definition of $B(x + y)$, so $b^{x+y} = \sup B(x + y) = \sup B(x) \sup B(y) = b^x b^y$.

9 Prove that there exist real numbers that are not algebraic.

The set of real numbers is uncountable, whereas the set of algebraic numbers is countable, as per problem seven. So there are "too many" real numbers for all of them to be algebraic.

10 Is the set of all irrational numbers countable? Why?

No, the irrational numbers are not countable.

Each rational number $r = p/q$ with $p, q \in \mathbb{Z}$, $q \ne 0$, is the root of $f(z) = qz - p$, implying that the rational numbers are a subset of the *countable* algebraic numbers, i.e. they're countable. So because the rational numbers are countable, the union of the rational and irrational numbers are the reals, and the reals are uncountable, then the irrational numbers are necessarily uncountable.