Math 508: Advanced Analysis
Homework 4

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1 Some limits

(a) Calculate $\lim_{n\to\infty} \frac{5n+17}{n+2}$

We first see that

$$\frac{5n+17}{n+2} = \frac{5n+17}{n+2}\left(\frac{1/n}{1/n}\right) = \frac{5+17/n}{1+2/n}$$

Since $1/n \to 0$ then $17/n \to 0$ and $2/n \to 0$. Therefore $5+17/n \to 5$ and $1+2/n \to 1$. Thus $\frac{1}{1+2/n} \to 1$ implying that $\frac{5+17/n}{1+2/n} \to 5$. Given the opening equation, this then final implies

$$\frac{5n+17}{n+2} \to 5$$

(b) Calculate $\lim_{n\to\infty} \frac{3n^2-2n+17}{n^2+21n+2}$

Similar to the previous part, we first see

$$\frac{3n^2-2n+17}{n^2+21n+2} = \frac{3-2/n+17/n^2}{1+21/n+2/n^2}$$

Since $1/n \to 0$, then $1/n^2 \to 0$. This implies $3-2/n+17/n^2 \to 3$ and $1+21/n+2/n^2 \to 1$. Hence $\frac{3-2/n+17/n^2}{1+21/n+2/n^2}$ converges to three and thus so does $\frac{3n^2-2n+17}{n^2+21n+2}$, as per the opening equation.

2 Calculate $\lim_{n\to\infty} \sqrt{n^2+n} - n$

We first note that

$$\begin{aligned}
\sqrt{n^2+n} - n &= \sqrt{n^2+n} - n\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} \\
&= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \\
&= \frac{n}{\sqrt{n^2+n}+n} \\
&= \frac{n}{\sqrt{n^2+n}+n}\left(\frac{1/n}{1/n}\right) \\
&= \frac{1}{\sqrt{1+\frac{1}{n}}+1}
\end{aligned}$$

We've seen in class that $\frac{1}{n} \to 0$ so that $1+\frac{1}{n} \to 1$. Since $1 < 1+\frac{1}{n}$, then $1 < \sqrt{1+\frac{1}{n}} < 1+\frac{1}{n}$, which means that $\sqrt{1+\frac{1}{n}} \to 1$ by the "squeeze theorem" in problem 6. It follows that $\sqrt{1+\frac{1}{n}}+1 \to 2$, from which which see

$$\frac{1}{\sqrt{1+\frac{1}{n}}+1} \to \frac{1}{2}$$

Hence, given the opening equation of this problem, we have $\sqrt{n^2+n} - n \to \frac{1}{2}$

3

Let $\{a_n > 0\}$ be a sequence of reals converging to a real $A > 0$. Then $B_{A/2}(A)$ contains all but finitely many points of $\{a_n\}$. Let a be the minimum value of the points not in $B_{A/2}(A)$. Note this point is positive since all a_n are positive. Thus setting $c = \min(a, A/2)/2$ we have that $a_n > c$ for all n.

4 Show that $\lim_{n\to\infty} \frac{c^n}{n!}$ for $c > 0$

Fix an integer m so that $m > 2c$. The for any $n > m$ we have that

$$0 < a_n = \frac{c}{n} a_{n-1} < \frac{c}{m} a_{n-1} < \frac{c}{2c} a_{n-1} = \frac{1}{2} a_{n-1} < \frac{1}{2^2} a_{n-2} < \frac{1}{2^3} a_{n-3} < \cdots < \frac{1}{2^{n-m}} a_m = 2^m a_m \frac{1}{2^n}$$

Thus $\lim_{n\to\infty} a_n \leq 2^m a_m \lim_{n\to\infty} \frac{1}{2^n} = 0$

5

Let $\{a_n\}$ and $\{b_n\}$ be real sequences.

(a) Show that $\limsup(a_n + b_n) \leq \limsup a_n + \limsup_n$

By definition we have

$$\lim_{N\to\infty} \sup (a_n + b_n) = \lim_{N\to\infty} (\sup(a_n + b_n) \mid n > N)$$

but for any integer N, $(\sup(a_n + b_n) \mid n > N) \leq (\sup(a_n) \mid n > N) + (\sup(b_n) \mid n > N)$ so that

$$\begin{aligned}
\limsup(a_n + b_n) &= \lim_{N\to\infty} (\sup(a_n + b_n) \mid n > N) \\
&\leq \lim_{N\to\infty} \Big((\sup(a_n) \mid n > N) + (\sup(b_n) \mid n > N) \Big) \\
&= \lim_{N\to\infty} (\sup(a_n) \mid n > N) + \lim_{N\to\infty} (\sup(b_n) \mid n > N) \\
&= \limsup(a_n) + \limsup(b_n)
\end{aligned}$$

(b) Give an explicit example where strict inequality can occur.

Let $\{a_n\}$ be $(-1)^n$ and $\{b_n\}$ be $(-1)^{n+1}$ so that $\{a_n\}$ has value $1, -1$ on even, odd indices, respectively, but $\{b_n\}$ has value $1, -1$ on odd, even indices, respectively. With these sequences

$$\lim_{n\to\infty} \sup (a_n + b_n) = 0$$

where as

$$\lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \sup b_n = 1 + 1 = 2$$

6 Prove a Squeeze Theorem

Let $\{r_n\}$, $\{s_n\}$, and $\{t_n\}$ be real sequences such that

$$r_n \leq s_n \leq t_n \ \forall n \tag{6.1}$$

Assume that both $t_n \to s$ and $r_n \to s$. Then for any $r > 0$, there is are integers N_1, N_2 such that $r_n, t_n \in B_{\frac{r}{3}}(s)$ for all $n \geq N$ where $N = \max(N_1, N_2)$. Therefore $|r_n - t_n| < \frac{2r}{3}$ which, combined with equation 6.1, implies $|s_n - t_n| < \frac{2r}{3}$ for each $n \geq N$. Therefore

$$|s_n - s| = |s_n - t_n + t_n - s| \leq |s_n - t_n| + |t_n - s| < \frac{2r}{3} + \frac{r}{3} = r$$

for each $n \geq N$. Hence $B_r(s)$ contains all but finitely many points in $\{s_n\}$, so that $s_n \to s$.

7

Let A be the set of natural numbers not divisible by three. Define

$$s_n := \frac{\#(A \cap \{1, \ldots, n\})}{n}$$

be the fraction of natural numbers from one to n which are not divisible by three. Thus defining the sequence $\{t_n\}$ where t_n is the fraction of natural numbers less than or equal to n that *are* divisibile by three, we have

$$s_n = 1 - t_n \tag{7.2}$$

for each n. Now $t_n = \frac{1}{n}\left\lfloor \frac{n}{3} \right\rfloor$, which implies $t_n \leq \frac{1}{3}$ for each n. Thus for any $\varepsilon > 0$ we have

$$\varepsilon > 0 = \frac{1}{3} - \frac{1}{3} \geq t_n - \frac{1}{3} = \left| t_n - \frac{1}{3} \right|$$

so that $t_n \to \frac{1}{3}$. Thus due to Equation 7.2, $s_n \to \frac{2}{3}$.

Repeat this for all natural numbers not multiples of two or three Let $\{t_n\}$ again be as above, the fraction of natural numbers less than or equal to n which are divisible by three. Define $\{r_n\}$ as the sequence where each r_n is the fraction of natural numbers less than or equal to n which are divisible by two. Thus we have $r_n = \frac{1}{n}\left\lfloor \frac{n}{2} \right\rfloor$, which with an argument nearly identical to the one for $\{t_n\}$ converging to $\frac{1}{3}$ above, we know $r_n \to \frac{1}{2}$. If, now, the sequence $\{a_n\}$ is such that each a_n is the fraction of natural numbers less than or equal to n which are not divisible by two or three, we have

$$a_n = (1 - t_n)(1 - r_n)$$

Hence a_n converges to $\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{2}\right) = \frac{1}{3}$ since t_n and r_n converge to $\frac{1}{3}$ and $\frac{1}{2}$, respectively.

8 Show the sequence of terms in Newton's method for solving $x^2 - A = 0$ converges to $\sqrt{A}$

Define $\{x_n\}$ recursively by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right)$$

We note that for the real-valued function $f(x) = \frac{1}{2}\left(x + \frac{A}{x}\right)$ we have $f'(x) = \frac{1}{2}\left(1 - \frac{A}{x^2}\right)$. Thus $f'(x) = 0$ when $x = \sqrt{A}$, implying that $f(\sqrt{A}) = \sqrt{A}$ is the minimum value of f. Since $\{x_n\}$ are just particular values of f, then $\{x_n\}$ is bounded below by $\sqrt{A}$.

So let's define a sequence $\{y_n\}$ by $y_n = x_n - \sqrt{A}$. Then we have

$$y_{n+1} = x_{n+1} - \sqrt{A} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right) - \sqrt{A} = \frac{1}{2}\left(y_n + \sqrt{A} + \frac{A}{y_n + \sqrt{A}}\right) - \sqrt{A} \leq \frac{1}{2}\left(y_n + \sqrt{A} + \frac{A}{\sqrt{A}}\right) - \sqrt{A} < \frac{y_n}{2}$$

This then indicates that $y_{n+1} < \frac{y_n}{2} < \cdots < \frac{y_1}{2^n}$ which in turn implies that $y_n \to 0$. Since we defined $y_n = x_n - \sqrt{A}$, then $x_n = y_n + \sqrt{A}$ yielding $x_n \to \sqrt{A}$.

9

This is not true. We know that the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. This implies that

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \tag{9.3}$$

also diverges since it's one less than the first. We use this fact to our advantage in finding a counter-example. Define the sequence $\{a_n\}$ by recursively by $a_0 = 0$ and

$$a_{n+1} = \begin{cases} a_n - \frac{1}{n+1} & a_n \geq 1 \\ a_n + \frac{1}{n+1} & a_n \leq 0 \\ \begin{cases} a_n + \frac{1}{n+1} & a_n - a_{n-1} > 0 \\ a_n - \frac{1}{n+1} & \text{otherwise} \end{cases} & \text{otherwise} \end{cases}$$

This sequence essentially bounces back and forth between zero and one, so is bounded. It also satisfies the property that $|a_n - a_{n-1}| < \frac{1}{n}$ since each point a_n differs by $\frac{1}{n+1}$ from the previous. Furthermore, we know once the sequence "hits" one and heads back to zero (and vice versa), that it will actually reach zero because of the divergence of 9.3 above; i.e. starting at zero but not being able to reach one (and vice versa) at any point in the sequence would imply convergence of 9.3.

10

For a real sequence $\{a_n\}$ define $\{c_n\}$ by $c_n = \frac{a_1 + \cdots + a_n}{n}$, i.e. $\{c_n\}$ is the sequence of partial averages of $\{a_n\}$.

(a) Find an example where $\{a_n\}$ doesn't converge, but $\{c_n\}$ does.

Let $a_n = (-1)^n$. Then $\{a_n\}$ doesn't converge, bouncing back and forth between 1 and -1, but $\{c_n\}$ is the sequence

$$-1, 0, \frac{-1}{3}, 0, \frac{-1}{5}, 0, \ldots$$

converging to zero.

(b) Show If $c_n \to A$ whenever $a_n \to A$

Let $a_n \to A$. For $\varepsilon > 0$ we can find an integer N such that

$$|a_n - A| < \frac{\varepsilon}{2} \tag{10.4}$$

Furthermore, because $\{a_n\}$ converges, it is bounded, so that we can find an integer M with

$$|a_n - A| < M \tag{10.5}$$

for all n. Thus for $n > N$ we have

$$c_n = \frac{a_1 + \cdots + a_N + a_{N+1} + \cdots + a_n}{n}$$

implying that

$$\begin{aligned}
|c_n - A| &= \left|\frac{a_1 + \cdots + a_N + a_{N+1} + \cdots + a_n}{n} - A\right| \\
&= \left|\frac{a_1 + \cdots + a_N + a_{N+1} + \cdots + a_n - nA}{n}\right| \\
&= \frac{1}{n}|(a_1 - A) + \cdots + (a_N - A) + (a_{N+1} - A) + \cdots + (a_n - A)| \\
&\le \frac{1}{n}\left(|a_1 - A| + \cdots + |a_N - A| + |a_{N+1} - A| + \cdots + |a_n - A|\right) \\
&< \frac{1}{n}\left(NM + (n-N)\frac{\varepsilon}{2}\right) \\
&< \frac{1}{n}\left(NM + \frac{n\varepsilon}{2}\right) \\
&= \frac{NM}{n} + \frac{\varepsilon}{2}
\end{aligned}$$

by making use of equations 10.4 and 10.5. Thus for any n with $n > \frac{2NM}{\varepsilon}$ we have

$$|c_n - A| < \frac{NM}{n} + \frac{\varepsilon}{2} < \frac{NM\varepsilon}{2NM} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \epsilon$$

implying the convergence of $\{c_n\}$ to A.

(c) If $a_k \ge 0$ and the averages converge, must $\{a_k\}$ be bounded?

No. For example, let $\{a_n\}$ be the sequence of partial sums of the harmonic series $\sum_{i=1}^{\infty} \frac{1}{i}$. We know that this series diverges, and so the sequence $\{a_n\}$ must be unbounded. However, in this case $\{c_n\}$ has

$$c_n = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{i} \le \frac{1}{n}\sum_{i=1}^{n}\frac{1}{n} = \frac{1}{n}$$

which implies $c_n \to 0$.