# Math 508: Advanced Analysis
## Homework 7

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# 1 Prove that smooth $f : [a, \infty) \to \mathbb{R}$ with bound first derivative is uniformly continuous.

Let $f : [a, \infty) \to \mathbb{R}$ be smooth with $M$ bound the first derivative, i.e. $|f'(x)| \leq M$. Let $\varepsilon > 0$ and set $\delta = \varepsilon/M$. Then for any $x, y \in [a, \infty)$, assuming without loss of generality that $x < y$, the mean value theorem informs us of a $c \in (x, y)$ such that

$$f(y) - f(x) = (y - x)f'(c)$$

Hence if $|y - x| < \delta$, we have

$$|f(y) - f(x)| = |(y - x)||f'(c)| \leq |(y - x)|M < \delta M = \varepsilon$$

so that $f$ is uniformly continuous.

# 2

## (a) Show that $\sin x$ is not a polynomial.

The function $\sin x$ is zero at $2\pi n$ for all integers $n$, i.e. it has infinitely many zeros. Polynomials have a finite amount of zeros, and so $\sin x$ cannot be a polynomial.

## (b) Show that $\sin x$ cannot be a rational function.

A rational function $p(x)/q(x)$ is zero if and only if $p(x)$ is zero. Therefore a rational function is zero at only finitely many points, and just as we saw in the previous part of the problem, this implies $\sin x$ cannot be a rational function.

## (c) If $f(t+1) = f(t)$ for all real $t$, and $f$ is not constant, show that $f$ is not a rational function.

By way of contradiction, assume that $f$ is a rational polynomial so that $f(t) = p(t)/q(t)$. Fixing $t_0 \in \mathbb{R}$ and by putting $g(t) = f(t) - f(t_0)$ we have

1. $g(t + 1) = f(t + 1) - f(t_0) = f(t) + f(t_0) = g(t)$ so that $g$ is periodic
2. $g(t) = p(t)/q(t) - f(t_0) = \frac{p(t) - f(t_0)q(t)}{q(t)}$ so that $g$ is rational, and
3. $g(t_0) = f(t_0) - f(t_0) = 0$ so that $g$ has a zero at $t_0$.

Putting the above three things together informs us that $g$ is a rational function with infinitely many zeros, but rational functions can only have a finite number of zeros; a contradiction.

## (d) Show that $e^x$ is not a rational function.

A rational function $f(x)$ has that

$$\lim_{x\to\infty} f(x) = \pm \lim_{x\to-\infty} f(x)$$

however

$$\lim_{x\to\infty} e^x = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} e^x = 0$$

## 3 Show that $\lim_{n\to\infty}(n+1)^{\frac{1}{7}} - n^{\frac{1}{7}} = 0$

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^{\frac{1}{7}}$. Since this is a polynomial, it's smooth on $\mathbb{R}$. For an integer $n$, the mean value theorem tells us that there is an $x \in (n, n+1)$ such that

$$(n+1)^{\frac{1}{7}} - n^{\frac{1}{7}} = (n+1-n)f'(x) = f'(x)$$

which in turn yields

$$(n+1)^{\frac{1}{7}} - n^{\frac{1}{7}} = \frac{1}{7}\left(\frac{1}{x^6}\right)^{\frac{1}{7}}$$

Hence

$$\lim_{n\to\infty}(n+1)^{\frac{1}{7}} - n^{\frac{1}{7}} = \lim_{x\to\infty}\frac{1}{7}\left(\frac{1}{x^6}\right)^{\frac{1}{7}} = 0$$

as desired.

## 4

Let $f : \mathbb{R} \to \mathbb{R}$ be smooth with $f(0) = 3$, $f(1) = 2$, and $f(3) = 8$. The Mean Value Theorem yields the existence of $c_1 \in (0, 1)$ and $c_2 \in (1, 3)$ such that

$$\begin{aligned}(1-0)f'(c_1) &= f(1) - f(0)\\ f'(c_1) &= 2-3\\ f'(c_1) &= -1\end{aligned}$$

and

$$\begin{aligned}(3-1)f'(c_2) &= f(3) - f(1)\\ 2f'(c_2) &= 8-2\\ 2f'(c_2) &= 6\\ f'(c_2) &= 3\end{aligned}$$

Because $f$ is smooth, $f'$ is continous since $f''$ is differentiable, and thus we can appeal to the Mean Value Theorem again to obtain a $c \in (c_1, c_2)$ such that

$$\begin{aligned}(c_2-c_1)f''(c) &= f'(c_2) - f'(c_1)\\ (c_2-c_1)f''(c) &= 3-(-1)\\ f''(c) &= \frac{4}{c_2-c_1}\end{aligned}$$

Since $c_2 > c_1 > 0$ this implies $f''(c) > 0$, as desired. Furthermore, because, more precisely, $3 > c_2 > 1 > c_1 > 0$ we have that $3 > c_2 - c_1$ so that $f''(c) > \frac{4}{3}$ according to the above equation. So let $M = \frac{4}{3}$.

## 5

By "a convex function $f$" we mean one for which every point of the graph of $f$ lies above all of its tangent points; i.e. one for which

$$f'(x)(y-x) + f(x) \leq f(y)$$

for all $x, y \in \mathbb{R}$.

## (a) Show that a smooth function $f$ is convex if $f''(x) \geq 0$ for all $x$.

Assume that the second derivative of $f$ is non-negative for any point of $\mathbb{R}$. For any reals $x, y$ with $x < y$ the Mean Value Theorem (MVT) gives us a $z \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(z) \tag{5.1}$$

Since $f$ is smooth, $f'$ is differentiable on $[x, z]$, and so the MVT gives us a $w \in (x, z)$ such that

$$\frac{f'(z) - f'(x)}{z - x} = f''(w)$$

Since the second derivative is non-negative, then so is the left hand side of the above equation. Since $z > x$ this implies $f'(z) \geq f'(x)$ which in light of Equation 5.1 implies $\frac{f(y)-f(x)}{y-x} \geq f'(x)$. This yields

$$f'(x)(y - x) + f(x) \leq f(y)$$

as desired for the convexity of $f$.

## (b) Prove that $v(x) \leq 0$ for all $0 \leq x \leq 1$ if $v''(x) > 0$ for $0 \leq x \leq 1$ and $v(0) = v(1) = 0$

Assume for later contradiction that there is a point $x \in (0, 1)$ with $v(x) > 0$. Then there exists a $c_1 \in (0, x)$ with

$$\frac{v(x) - v(0)}{x - 0} = v'(c_1)$$

by the MVT so that $\frac{v(x)}{x} = v'(c_1)$ which implies $v'(c_1) > 0$ since $v(x) > 0$. Furthermore there exists an $c_2 \in (x, 1)$ where

$$\frac{v(1) - v(x)}{1 - x} = v'(c_2)$$

so that $\frac{-v(x)}{1-x} = v'(c_2)$. Since $x < 1$ and $v(x) > 0$, then $v'(c_2) < 0$. Once more, the MVT tells us there exists a $c \in (c_1, c_2)$ such that

$$\frac{v'(c_2) - v'(c_1)}{c_2 - c_1} = v''(c)$$

but since we've seen that $v'(c_1) > 0$, $v'(c_2) < 0$, and because $c_2 > c_1$, the above equation yields $v''(c) \leq 0$. This contradicts the fact that $v''(x)$ for $0 \leq x \leq 1$. Hence there is no point $x \in (0, 1)$ with $v(x) > 0$, and therefore $v(x) \leq 0$ for all $x \in [0, 1]$.

## (c) Prove that $e^x$ is convex.

The second derivative of $e^x$ is $e^x$, which is always positive. By the first part of this problem we know that $e^x$ is convex.

## (d) Prove that $e^x \geq 1 + x$ for all $x$

Since the previous part of this problem showed $e^x$ is convex, then for any $x, y$ we have

$$e^x \geq \left(\frac{d}{dy} e^y\right)(x - y) + e^y = e^y(x - y) + e^y$$

Thus, letting $y = 0$, we get $e^x \geq x + 1$ for all $x$.

# 6

## (a) What constraints are on $c$ and $d$ so that $p(x) = x^3 + cx + d$ has three distinct real roots?

If $p(x) = x^3 + cx + d$ were to have three distinct real roots, then there would exist real $x_1 < x_2$ where $p(x_1) > 0$ is a local maximum and $p(x_2) < 0$ is a local minimum. Since $\lim_{x\to-\infty} p(x) = -\infty$ and $\lim_{x\to\infty} p(x) = \infty$, then we can find $x_0, x_3 \in \mathbb{R}$ with $x_0 < x_1$, $x_3 > x_2$, $p(x_0) < 0$, and $p(x_3) > 0$. Thus the intermediate value theorem implies, since $x_1 < x_2$, $p(x_1) > 0$, and $p(x_2) < 0$, the existence of $c_1, c_2, c_3 \in \mathbb{R}$ where $x_0 < c_1 < x_1 < c_2 < x_2 < c_3 < x_3$ and $p(c_1) = p(c_2) = p(c_3) = 0$. Thus it is indeed possible for there to exist three distinct real roots.

We have that $p'(x) = 3x^2 + c$ so that $x = \pm\sqrt{\frac{-c}{3}}$ when $p'(x) = 0$. Hence in order for there to be three real roots, $c$ must be less than zero. Since $x = \pm\sqrt{\frac{-c}{3}}$ are the two local maximum and minimum, then $p(\sqrt{\frac{-c}{3}}) < 0$ so that

$$\begin{aligned}
p(\sqrt{\frac{-c}{3}}) &< 0 \\
\frac{-c}{3}\sqrt{\frac{-c}{3}} + c\sqrt{\frac{-c}{3}} + d &< 0 \\
d &< \frac{c}{3}\sqrt{\frac{-c}{3}} - c\sqrt{\frac{-c}{3}} \\
d &< c\sqrt{\frac{-c}{3}}\left(\frac{1}{3} - 1\right) \\
d &< \frac{-2c}{3}\sqrt{\frac{-c}{3}}
\end{aligned}$$

At this point, since we have $c < 0$, then the right hand side of the above inequality is positive so that

$$\begin{aligned}
d^2 &< \left(\frac{-2c}{3}\sqrt{\frac{-c}{3}}\right)^2 \\
d^2 &< \frac{-4c^2}{9}\left(\frac{-c}{3}\right) \\
d^2 &< \frac{4c^3}{27}
\end{aligned}$$

and this is the constraint on $d$.

## (b) Generalize above to $p(x) = ax^3 + bx^2 + cx + d$

# 7

# 8

## (a)

**(b)**

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**(c)**

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**(d)**

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