Math 509: Advanced Analysis
Homework 1

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January 24, 2015
http://coursework.tylerlogic.com/courses/upenn/math509/homework01

1 Lecture Notes Problem 1

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{xy}{x^2+y^2} & \text{otherwise} \end{cases}$$

This then yields the following partial derivatives $f_x : \mathbb{R}^2 \to \mathbb{R}$ and $f_y : \mathbb{R}^2 \to \mathbb{R}$ for nonzero points

$$f_x(x,y) = \frac{y}{x^2+y^2} - \frac{2x^2y}{(x^2+y^2)^2}$$

and

$$f_y(x,y) = \frac{x}{x^2+y^2} - \frac{2xy^2}{(x^2+y^2)^2}$$

Thus there are no points in $\mathbb{R}^2 - (0,0)$ for which f_x and f_y don't exist. It remains to be proven that the partial derivatives exist at the origin. Since the following expression evaluates to zero:

$$f_x = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$$

then the partial derivative f_x exists at $(0,0)$. A symmetrical argument proves the existence of f_y at $(0,0)$.

2 Lecture Notes Problem 2

Let U be a convex set in $\mathbb{R}^m$ and $f : U \to \mathbb{R}^n$ a differentiable map such that there exists a real M with $|f'(x)| \leq M$ for all $x \in U$. Let $p, q \in U$ and define two maps $\phi : (0,1) \to U$, $g : (0,1) \to \mathbb{R}^n$ where $(0,1) \subset \mathbb{R}$ is an open interval, by

$$\phi(t) = tp + (1-t)q \qquad \text{and} \qquad g(t) = f(\phi(t))$$

Note that g is well defined because the range of ϕ is a subset of U due to U being convex. So taking the derivative of g we get $g'(t) = f'(\phi(t))\phi'(t)$ by the chain rule so that $g'(t) = f'(\phi(t))(p-q)$ due to the fact that ϕ the function for the points on the line segment connecting p and q. Taking the norm of both sides then yields

$$|g'(t)| = |f'(\phi(t))||p-q| \leq M|p-q| \tag{2.1}$$

Now Rudin Theorem 5.19 tells us that $|g(1) - g(0)| \leq (1-0)|g'(t)|$ for all $t \in [0,1]$, which implies

$$|f(p) - f(q)| = |f(\phi(1)) - f(\phi(0))| = |g(1) - g(0)| \leq |g'(t)| \tag{2.2}$$

Combining equations 2.1 and 2.2 gives to us the desired result:

$$|f(p) - f(q)| \leq M|p-q|$$

3 Lecture Notes Problem 4

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = 2y^2 - x(x-1)^2$$

Then the partial derivatives of f are

$$\begin{aligned} f_x &= -(x-1)^2 - 2x(x-1) = \\ f_y &= 4y \end{aligned}$$

so that f_x is zero at $x = \frac{1}{3}, 1$ and f_y is zero when $y = 0$. Thus the critical points are $(\frac{1}{3}, 0)$ and $(1,0)$. Finally, because the Hessian matrix is

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 4-6x & 0 \\ 0 & 4 \end{pmatrix}$$

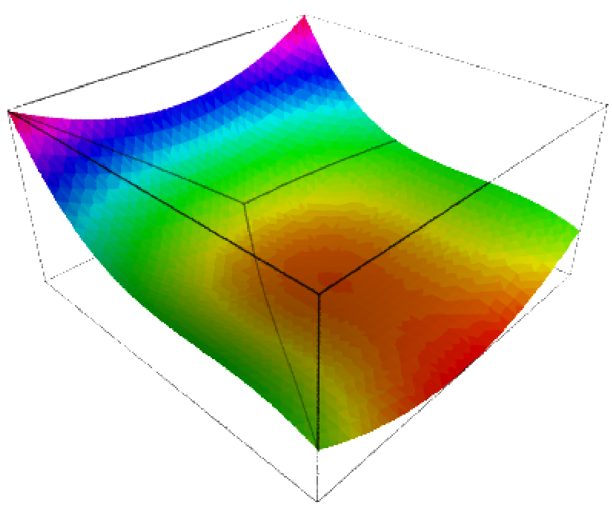
then

$$\det(H(\frac{1}{3}, 0)) = \det \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} = 8$$

and

$$\det(H(1,0)) = \det \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix} = -8$$

implying that the point $(\frac{1}{3}, 0)$ is a local minimum and $(1,0)$ is a saddle point.
We see this function graphed with the following sketch:



4 Lecture Notes Problem 5

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = ax^2 + 2bxy + cy^2$$

Then the partial derivatives of f are

$$\begin{aligned} f_x &= 2ax + 2by \\ f_y &= 2bx + 2cy \end{aligned}$$

Thus we get that $f_x = 0$ whenever $ax = -by$ and $f_y = 0$ whenever $bx = -cy$. Hence the only critical point is $(0, 0)$. The Hessian matrix

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2a & 2b \\ 2b & 2c \end{pmatrix}$$

is completely independent of x and y. So because $\det(H) = 4ac - 4b^2$ then the critical point $(0, 0)$ will be a local minimum if $ac > b^2$ and $a, c > 0$, a local maximum if $ac > b^2$ and $a, c < 0$, saddle point if $ac < b^2$, nondegenerate if $ac - b^2 \neq 0$, and degenerate if $ac = b^2$.

5 Lecture Notes Problem 6

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = Real((x_i y)^3) = x^3 - 3xy^2$$

Then the partial derivatives of f are

$$\begin{aligned} f_x &= 3x^2 - 3y^2 \\ f_y &= -6xy \end{aligned}$$

so that the only critical point is $(0, 0)$. Finally, because the Hessian matrix is

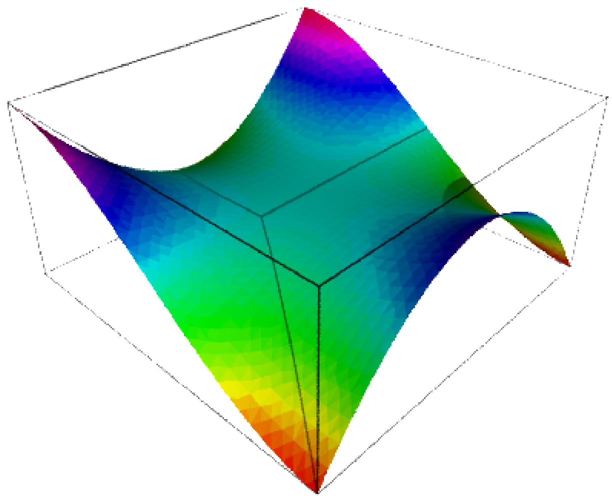
$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6x & -6y \\ -6y & -6x \end{pmatrix}$$

then

$$\det(H(0, 0)) = \det \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 8$$

implying that the point $(0, 0)$ is a nondegenerate critical point.

We see this function graphed with the following sketch:



Given this graph, a monkey saddle is a saddle with depressions for three legs. . . or in the case of a monkey, two legs and a tail.

6 Lecture Notes Problem 7

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = Real((x_i y)^3) = x^2 y^2$$

Then the partial derivatives of f are

$$\begin{aligned} f_x &= 2xy^2 \\ f_y &= 2x^2 y \end{aligned}$$

so that all points of the form $(0, y)$ or $(x, 0)$ are critical points. Because the Hessian matrix is

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2y^2 & 4xy \\ 4xy & 2x^2 \end{pmatrix}$$

then

$$\det(H(0, y)) = \det \begin{pmatrix} 2y^2 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

and

$$\det(H(x, 0)) = \det \begin{pmatrix} 0 & 0 \\ 0 & 2x^2 \end{pmatrix} = 0$$

Thus all critical points are nondegenerate.

We see this function graphed with the following sketch: