Math 509: Advanced Analysis
Homework 2

Lawrence Tyler Rush
<me@tylerlogic.com>

February 1, 2015
http://coursework.tylerlogic.com/courses/upenn/math509/homework02

1 Provide a complete proof of the Chain Rule

Let $f : R^m \to R^n$ and $g : R^n \to R^p$ be differentiable functions. Furthermore, let $x_0 \in R^n$, $y_0 \in R^m$, and $z_0 \in R^p$ such that $f(x_0) = y_0$ and $g(y_0) = z_0$. For ease of notation, we asssume that, without loss of generality, the points $x_0 \in R^m$, $y_0 \in R^n$ and $z_0 \in R^p$ are the origins, respectively, of R^m, R^n, and R^p. We may safely assume this because this situation can be obtained with a simple translation of the functions f and g, which leaves there differentiability, namely the limit formula that makes them differentiable, unaffected.

Now define the linear maps $L = f'(x_0)$ and $M = g'(y_0)$ to be the derivatives of f and g, respectively, at x_0 and y_0. Thus we have

$$\frac{f(x) - L(x)}{|x|} = \frac{f(x_0 + x) - f(x_0) - L(x)}{|x|} \to 0 \quad \text{as} \quad x \to 0$$

and

$$\frac{g(y) - M(y)}{|y|} = \frac{g(y_0 + y) - g(y_0) - M(y)}{|y|} \to 0 \quad \text{as} \quad y \to 0$$

via the combination of the differentiability of f and g and our assumption that x_0, $f(x_0) = y_0$, and $g(y_0) = z_0$ are all at the origin of their respective spaces. These two equations then yield the following sequence of equations for sufficiently small $x \in R^m$

$$\begin{aligned}
|gf(x) - ML(x)| &= |gf(x) - \big(Mf(x) - Mf(x)\big) - ML(x)| \\
&= |gf(x) - Mf(x) + Mf(x) - ML(x)| \\
&\leq |g - M||f(x)| + |M||f(x) - L(x)| \\
&< \varepsilon|f(x)| + |M|(\varepsilon|x|)
\end{aligned}$$

Dividing the last line by $|x|$ then yields

$$\frac{|gf(x) - ML(x)|}{|x|} < \varepsilon\frac{|f(x)|}{|x|} + \varepsilon|M|$$

Hence if we can show that $\frac{|f(x)|}{|x|}$ is bounded as $|x| \to 0$, then the previous inequality implies that $\frac{|gf(x)-ML(x)|}{|x|}$ can be arbitrarily bounded as $|x| \to 0$. Such a property on $\frac{|gf(x)-ML(x)|}{|x|}$ proves our theorem since

$$\frac{|gf(x_0 + x) - gf(x_0) - ML(x)|}{|x|} = \frac{|gf(x) - ML(x)|}{|x|}$$

as we assumed x_0 is the origin.

So to, finally, prove the boundedness of $|f(x)|/|x|$ we see that

$$\frac{|f(x)|}{|x|} = \frac{|f(x) + \big(L(x) - L(x)\big)|}{|x|} \leq \frac{|L(x)| + |f(x) - L(x)|}{|x|} = \frac{|L(x)|}{|x|} + \frac{|f(x) - L(x)|}{|x|}$$

but $|L(x)|/|x| < |L|$ and $\frac{|f(x)-L(x)|}{|x|} \to 0$ as $|x| \to 0$. Hence $\frac{|f(x)|}{|x|}$ is bounded as $|x| \to 0$ as desired.

2 Provide a complete proof of the equality mixed partials (for $m = 2$)

Let U be an open set in R^2 and $f : U \to R$ a function whose partial derivatives of orders one and two exist and are continuous on U. So for arbitrary $h \in R$, define the function $g_h : U \to R$ by

$$g_h(x, y) = f(x + h, y) - f(x, y)$$

and then define $E : U \to R$ by

$$E(h, k) = g_h(x, y + k) - g_h(x, y)$$

Due to the definition of g_h we can apply the mean value theorem (MVT) to E twice, but in two different ways:

1. We can apply the MVT to g_h in E to get
$$E(h,k) = k\frac{\partial}{\partial y}g_h(x, y+t_1k)$$
for some $0 \le t_1 \le 1$. We next expand g_h in the result to obtain
$$E(h,k) = k\frac{\partial}{\partial y}\left(f(x+h, y+t_1k) - f(x, y+t_1k)\right)$$
and finally apply the MVT once more now to f to get
$$E(h,k) = hk\frac{\partial^2}{\partial y \partial x}f(x+s_1h, y+t_1k) \tag{2.1}$$
for some $0 \le s_1 \le 1$.

2. Alternatively, we can first expand both g_h components of E obtaining
$$E(h,k) = \Big(f(x+h, y+k) - f(x, y+k)\Big) - \Big(f(x+h, y) - f(x,y)\Big)$$
and then apply the MVT first to the resulting expansion of the first component and then a second time to the resulting expansion of the second component to get
$$\begin{aligned} E(h,k) &= \Big(f(x+h, y+k) - f(x, y+k)\Big) - \Big(f(x+h, y) - f(x,y)\Big) \\ &= h\frac{\partial}{\partial x}f(x+s_2h, y+k) - h\frac{\partial}{\partial x}f(x+s_2h, y) \\ &= h\frac{\partial}{\partial x}\left(f(x+s_2h, y+k) - f(x+s_2h, y)\right) \end{aligned}$$
for some $0 \le s_2 \le 1$. We then apply the MVT once more yielding
$$E(h,k) = h\frac{\partial}{\partial x}\left(k\frac{\partial}{\partial y}f(x+s_2h, y+t_2k)\right) = hk\frac{\partial^2}{\partial x \partial y}f(x+s_2h, y+t_2k) \tag{2.2}$$
for some $0 \le t_2 \le 1$.

Therefore equations 2.1 and 2.2 yield
$$\frac{\partial^2}{\partial y \partial x}f(x+s_1h, y+t_1k) = \frac{E(h,k)}{hk} = \frac{\partial^2}{\partial x \partial y}f(x+s_2h, y+t_2k) \tag{2.3}$$

Now in light of the definition of E, we see that E is also continuous in both h and k so that letting $h \to 0$ followed by letting $k \to 0$ will have the same value as letting $k \to 0$ followed by letting $h \to 0$. Thus we can let $h \to 0$ and $k \to 0$, safely in either order, in equation 2.3 to obtain
$$\frac{\partial^2}{\partial y \partial x}f(x,y) = \frac{\partial^2}{\partial x \partial y}f(x,y)$$
as desired.

3 Problem 18 from Slides

Let $f : R^2 \to R$ be a homogeneous function of degree n.

(a) If f is also differentiable, show $f = xf_x + yf_y$

Let f be differentiable. By defiinition, we have that $f(tx, ty) = t^n f(x,y)$ for any t. Taking the derivative of each side of with respect to t gives us
$$xf_x(tx, ty) + yf_y(tx, ty) = nt^{n-1}f(x,y) \tag{3.4}$$
Since this is true for any $t \in R$, setting $t = 1$ yields $xf_x(x,y) + yf_y(x,y) = nf(x,y)$, as desired.

(b) If f is twice differentiable, show $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = n(n-1)f$

By again taking the derivative with respect to t of both sides of 3.4 we get

$$x\big(xf_{xx}(tx,ty) + yf_{xy}(tx,ty)\big) + y\big(xf_{yx}(tx,ty) + yf_{yy}(tx,ty)\big) = n(n-1)t^{n-2}f(x,y)$$

which becomes

$$x^2 f_{xx}(tx,ty) + 2xyf_{xy}(tx,ty) + y^2 f_{yy}(tx,ty) = n(n-1)t^{n-2}f(x,y)$$

upon expanding and combining like terms. Thus by, once more, evaluating this formula at $t = 1$ we obtain the desired formula.

(c) Test the above parts of the problem on $f(x,y) = xy/(x+y)$

By the following, we see that f is homogeneous of degree 1:

$$f(tx,ty) = \frac{t^2xy}{t(x+y)} = t\frac{xy}{x+y} = tf(x,y)$$

so we need to confirm the formulas from the previous part of the problem. For $xf_x + yf_y$ we get

$$\begin{aligned}
xf_x + yf_y &= x\left(\frac{y}{x+y} - \frac{xy}{(x+y)^2}\right) + y\left(\frac{x}{x+y} - \frac{xy}{(x+y)^2}\right)\\
&= 2\frac{xy}{x+y} - \frac{x^2y}{(x+y)^2} - \frac{xy^2}{(x+y)^2}\\
&= 2\frac{xy}{x+y} - \frac{x^2y + xy^2}{(x+y)^2}\\
&= 2\frac{xy}{x+y} - xy\frac{x+y}{(x+y)^2}\\
&= 2\frac{xy}{x+y} - \frac{xy}{x+y}\\
&= \frac{xy}{x+y}\\
&= f
\end{aligned}$$

as desired.

Now we need to confirm the second formula. To start, we have

$$\begin{aligned}
f_{xx} &= 2\frac{xy}{(x+y)^3} - 2\frac{y}{(x+y)^2}\\
f_{xy} &= 2\frac{xy}{(x+y)^3}\\
f_{yy} &= 2\frac{xy}{(x+y)^3} - 2\frac{x}{(x+y)^2}
\end{aligned}$$

This in turns leads to

$$\begin{aligned}
x^2f_{xx} + 2xyf_{xy} + y^2f_{yy} &= x^2\left(2\frac{xy}{(x+y)^3} - 2\frac{y}{(x+y)^2}\right) + 2xy\left(2\frac{xy}{(x+y)^3}\right) + y^2\left(2\frac{xy}{(x+y)^3} - 2\frac{x}{(x+y)^2}\right)\\
&= \frac{2x^3y}{(x+y)^3} - \frac{2x^2y}{(x+y)^2} + \frac{4x^2y^2}{(x+y)^3} + \frac{2xy^3}{(x+y)^3} - \frac{2xy^2}{(x+y)^2}\\
&= \frac{2x^3y}{(x+y)^3} - \frac{2x^3y + 2x^2y^2}{(x+y)^3} + \frac{4x^2y^2}{(x+y)^3} + \frac{2xy^3}{(x+y)^3} - \frac{2x^2y^2 + 2xy^3}{(x+y)^3}\\
&= \frac{2x^3y - 2x^3y - 2x^2y^2 + 4x^2y^2 + 2xy^3 - 2x^2y^2 - 2xy^3}{(x+y)^3}\\
&= 0
\end{aligned}$$

as desired, since $n(n-1)f$ is zero for $n = 1$.

4 Problem 19 from Slides

Define $u(x, y) = x^2 - y^2$ and $v(x, y) = y^2 - x^2$, then z is simply $f(u(x, y), v(x, y))$ from which we get

$$z_x = u_x f_u - u_x f_v = 2x f_u - 2x f_v \qquad \text{and} \qquad z_y = u_y f_u - u_y f_v = -2y f_u + 2y f_v$$

so that

$$y z_x + x z_y = (2xy f_u - 2xy f_v) + (-2xy f_u + 2xy f_v) = 2xy f_u - 2xy f_u - 2xy f_v + 2xy f_v = 0$$

as desired.

5

Problem: Define $f : R \to R$ by

$$f(x) = \begin{cases} 0 & x = 0 \\ x^2 \sin(1/x) & \text{otherwise} \end{cases}$$

Prove that f is differentiable for all x, including $x = 0$, but the derivative $f'(x)$ is not continuous at $x = 0$.