Math 509: Advanced Analysis
Homework 6

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To aide in our proofs, let's create an equivalent definition of Riemann Integrable.

Definition 1. A function $f : A \to R$ for $A \subset R^n$ is *Riemann Integrable* if for all real $\varepsilon > 0$ there exists a partition P of A such that

$$U(f,P) - L(f,P) < \varepsilon$$

Note this is simply Theorem 3-3 of Spivak.

1 Problem 3 from Slides

Problem:
Prove that a continuous function $f : A \to R$ where $A \subset R^n$ and $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is Riemann Integrable.

Solution:
We will make use of Definition 1 for an easier proof. So let $\varepsilon > 0$. Define $\eta > 0$ so that

$$\eta \prod_i (b_i - a_i) < \varepsilon \tag{1.1}$$

Because A is compact and f continuous on A, then f is uniformly continuous on A. Hence there is a $\delta > 0$ such that $|x - y| < \delta$ implies

$$|f(x) - f(y)| < \eta \tag{1.2}$$

for all $x, y \in A$. Now define a partition $P = (P_1, \ldots, P_n)$ of A by $P_i = \{a_i, a_i + k_i, a_i + 2k_i\}$ where $k_i = \frac{b_i - a_i}{m}$ and m is an integer chosen so that $b_i - a_i < m\frac{\delta}{\sqrt{n}}$ for all i. Defining P in this way means that any two points x, y contained in the same rectangle $S \in P$ will have

$$|x - y| < \sqrt{\left(\frac{\delta}{\sqrt{n}}\right)^2 + \cdots + \left(\frac{\delta}{\sqrt{n}}\right)^2} = \sqrt{n\left(\frac{\delta}{\sqrt{n}}\right)^2} = \delta$$

Hence, by equation 1.2 we have

$$M_S(f) - m_S(f) < \eta \tag{1.3}$$

due to f attaining its maximum and minimum value on the compact set S. Finally, through the use of equations 1.1 and 1.3 we obtain

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{S\in P} M_S(f)\,\mathrm{v}(S) - \sum_{S\in P} m_S(f)\,\mathrm{v}(S) \\
&= \sum_{S\in P} (M_S(f) - m_S(f))\,\mathrm{v}(S) \\
&< \eta \sum_{S\in P} \mathrm{v}(S) \\
&= \eta \prod_i (b_i - a_i) \\
&< \varepsilon
\end{aligned}$$

as desired.

2 Problem 4 from Slides

Problem:
Let $A \subset R^n$ and denote it by $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Show that if $f : A \to R$ has only finitely many points of discontinuity, then it is Riemann integrable.

Solution:
Let $\varepsilon > 0$. We will show the existence of a partition P of A such that $U(f,P) - L(f,P) < \varepsilon$. To ease notation and therefore our proof, we define $M = |f(x)|$ and define $\eta > 0$ so that

$$\eta\,\mathrm{v}(A) < \varepsilon/2 \tag{2.4}$$

both of which we will use shortly.

Secluding Discontinuities. Let $E = \{x_1, \dots, x_k\}$ be the points of discontinuity of f denoting each x_i by $x_i = (x_{i1}, x_{i2}, \dots, x_{in})$. Then define k closed squares $\mathcal{A} = \{A_1, \dots, A_k\}$ by setting the length of the sides of each square to ℓ where ℓ is chosen so that

$$\ell < \frac{1}{n} \min_{i,j} \{|x_i - x_j|\}$$

and

$$2Mk\ell^n < \varepsilon/2 \tag{2.5}$$

and then putting each $A_i = [x_{i1} - \frac{\ell}{2}, x_{i1} + \frac{\ell}{2}] \times \cdots \times [x_{in} - \frac{\ell}{2}, x_{in} + \frac{\ell}{2}]$. By defining A_i this way, the first restriction on ℓ above ensures $A_1, \dots, A_k$ are mutually exclusive. The second restriction above simply allows an important bound we'll see shortly.

Partitioning $\mathcal{A}$. Since E is covered by $\mathcal{A}$ and each point of E in the interior of an element of $\mathcal{A}$, then $K = A - \cup_i A_i^\circ$, where A_i° is the interior of A_i, is compact and has no intersection with E. Since f is continuous on A, it is continuous on K, and thus uniformly continuous on compact K. Hence we can find a $\delta > 0$ such that

$$|f(s) - f(t)| < \eta \tag{2.6}$$

whenever $|s - t| < \delta$ for any $s, t \in K$. Now define a partition $P = (P_1, \dots, P_n)$ of A by

$$P_i = \{a_i, a_i + r_i, a_i + 2r_i, \dots, a_i + (z-1)r_i, b_i\} \cup \left\{x_{1i} \pm \frac{\ell}{2}, \dots, x_{mi} \pm \frac{\ell}{2}\right\} - \bigcup_j \left(x_{ji} - \frac{\ell}{2}, x_{ji} + \frac{\ell}{2}\right)$$

where $r_i = \frac{b_i - a_i}{z}$ and z is an integer chosen so that $\frac{b_i - a_i}{z} < \delta$ for all i. Defining P_i in this manner ensures that no point of $\left(x_{ji} - \frac{\ell}{2}, x_{ji} + \frac{\ell}{2}\right)$, for any j, is contained in P_i and that all points outside of those intervals are at most within a distance of δ of each other. This restriction on the distance yields

$$M_S(f) - m_S(f) < \eta \tag{2.7}$$

for any $S \in P - \mathcal{A}$ due to equation 2.6.

Conclusion. Given our definition of P and because $|M_S(f) - m_S(f)| \le 2M$ for any $S \in P$, we have the following sequence of equations allotted to us by equations 2.4, 2.5, and 2.7

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{S \in P} |M_S(f) - m_S(f)| \operatorname{v}(S) \\
&= \sum_{S \in P - \mathcal{A}} |M_S(f) - m_S(f)| \operatorname{v}(S) + \sum_{S \in \mathcal{A}} |M_S(f) - m_S(f)| \operatorname{v}(S) \\
&< \sum_{S \in P - \mathcal{A}} \eta \operatorname{v}(S) + \sum_{S \in \mathcal{A}} |M_S(f) - m_S(f)| \operatorname{v}(S) \\
&\le \sum_{S \in P - \mathcal{A}} \eta \operatorname{v}(S) + 2M \sum_{S \in \mathcal{A}} \operatorname{v}(S) \\
&\le \sum_{S \in P - \mathcal{A}} \eta \operatorname{v}(S) + 2M \sum_{S \in \mathcal{A}} \ell^n \\
&= \eta \sum_{S \in P - \mathcal{A}} \operatorname{v}(S) + 2Mk\ell^n \\
&\le \eta \operatorname{v}(A) + 2Mk\ell^n \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon
\end{aligned}$$

yielding our desired bound.

3 Problem 5 from Slides

Let $f : A \to R$ and $g : A \to R$ where $A \subset R^n$ and $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$ both be Riemann Integrable.

(a) Show that $f + g$ is Riemann Integrable

We first note that because for any bounded set S, M_S and m_S are just functions whose outputs are supremums and infimums, respectively, then we have that

$$M_S(f) + M_S(g) \geq M_S(f + g) \tag{3.8}$$

and

$$m_S(f) + m_S(g) \leq m_S(f + g) \tag{3.9}$$

implying that

$$(M_S(f) + M_S(g)) - (m_S(f) + m_S(g)) \geq M_S(f + g) - m_S(f + g) \tag{3.10}$$

This we state for later use.

So now let $\varepsilon > 0$. Since f and g are Riemann Integrable we can then find partitions P_1 and P_2 of A such that $U(f, P_1) - L(f, P_1) < \varepsilon/2$ and $U(g, P_2) - L(g, P_2) < \varepsilon/2$. Putting $P = P_1 \cup P_2$ refines both P_1 and P_2 simultaneously, thus yielding $U(f, P) - L(f, P) < \varepsilon/2$ and $U(g, P) - L(g, P) < \varepsilon/2$. Adding these inequalities gives us

$$(U(f, P) - L(f, P)) + (U(g, P) - L(g, P)) < \varepsilon$$

so that through application of equation 3.10 we get

$$\begin{aligned}
\varepsilon &> (U(f, P) - L(f, P)) + (U(g, P) - L(g, P)) \\
&= \sum_{S \in P} (M_S(f) - m_S(f))\, \mathrm{v}(S) + \sum_{S \in P} (M_S(g) - m_S(g))\, \mathrm{v}(S) \\
&= \sum_{S \in P} ((M_S(f) + M_S(g)) - (m_S(f) + m_S(g)))\, \mathrm{v}(S) \\
&\geq \sum_{S \in P} (M_S(f + g) - m_S(f + g))\, \mathrm{v}(S) \\
&= U(f + g, P) - L(f + g, P)
\end{aligned}$$

which implies that $f + g$ is Riemann Integrable.

(b) Show that $\int_A (f + g) = \int_A f + \int_A g$

For any function $h : A \to R$, partition P of A, and $S \in P$ we have $m_S(h) \leq M_S(h)$. Thus equations 3.8 and 3.9 tell us that

$$m_S(f) + m_S(g) \leq m_S(f + g) \leq M_S(f + g) \leq M_S(f) + m_S(g)$$

for f and g. Hence for any partition P of A,

$$\sum_{S \in P} (m_S(f) + m_S(g))\, \mathrm{v}(S) \leq \sum_{S \in P} m_S(f + g)\, \mathrm{v}(S) \leq \sum_{S \in P} M_S(f + g)\, \mathrm{v}(S) \leq \sum_{S \in P} (M_S(f) + M_S(g))\, \mathrm{v}(S)$$

which implies

$$L(f, P) + L(g, P) \leq L(f + g, P) \leq U(f + g, P) \leq U(f, P) + U(g, P)$$

Since f and g are Riemann Integrable, then $L(f, P) + L(g, P)$ and $U(f, P) + U(g, P)$ can be brought arbitrarily close to each other. Thus the above inequality implies that all of $L(f + g, P)$, $L(f, P) + L(g, P)$, $U(f, P) + U(g, P)$, and $U(f + g, P)$ can be made arbitrarily close to each other by choosing an appropriate partition. Hence

$$\int_A (f + g) = \int_A f + \int_A g$$

(c) For constant c, show that cf is Riemann Integrable

Let $\varepsilon > 0$. Since f is Riemann Integrable, we can find a partition P of A such that

$$U(f,P) - L(f,P) < \frac{\varepsilon}{c}$$

Since $U(cf,P) = cU(f,P)$ and $L(cf,P) = cL(f,P)$, then

$$U(cf,P) - L(cf,P) = c\left(U(f,P) - L(f,P)\right) < c\left(\frac{\varepsilon}{c}\right) = \varepsilon$$

so that cf is Riemann Integrable.

(d) For constant c, show that $\int_A cf = c\int_A f$

Since $U(cf,P) = cU(f,P)$, then the Riemann Integrability of f and cf implies

$$\int_A cf = \inf_P U(cf,P) = \inf_P \left(cU(f,P)\right) = c\inf_P U(f,P) = c\int_A f$$

as desired.

4 Problem 6 from Slides

Problem:
Show that we can use open rectangles instead of closed rectangles in the definition of "measure zero" and the sets that have measure zero will remain unchanged

Solution:
There is nothing really to prove to show that the open rectangle definition implies the closed rectangle definition since a countable set of open rectangles is a subset of set of those rectangles' closures, but have the same volume.

To prove that the closed rectangle definition implies the open rectangle definition, let $A \subset R^n$ be a set of measure zero using the closed rectangle definition. Let $\varepsilon > 0$. Then we can find a countable collection of closed sets $\{V_i\}$ that covers A such that

$$\sum_i \mathrm{v}(V_i) < \frac{\varepsilon}{2}$$

and denote each V_i by $[a_{i1}, b_{i1}] \times \cdots \times [a_{in}, b_{in}]$. Choose $r > 0$ so that $(1+r)^n < 2$ and define a countable collection of open sets $\{U_i\}$ by

$$U_i = \left(a_{i1} - \frac{r}{2}, b_{i1} + \frac{r}{2}\right) \times \cdots \times \left(a_{in} - \frac{r}{2}, b_{in} + \frac{r}{2}\right)$$

for each i. Then with this definition we have $V_i \subset U_i$ indicating that $\{U_i\}$ is a cover of A by open rectangles. But furthermore, the volume of each open rectangle is

$$\mathrm{v}(U_i) = \prod_j \left(\left(b_{ij} + \frac{r}{2}\right) - \left(a_{ij} - \frac{r}{2}\right)\right) = \prod_j \left(\left(b_{ij} - a_{ij}\right)(1+r)\right) = (1+r)^n \prod_j \left(b_{ij} - a_{ij}\right) < 2\,\mathrm{v}(V_i)$$

implying that the volume of the entire collection is

$$\sum_i \mathrm{v}(U_i) < \sum_i 2\,\mathrm{v}(V_i) = 2\sum_i \mathrm{v}(V_i) < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon$$

which reveals that A has measure zero according to the open rectangle definition, as well.

5 Problem 8 from Slides

Problem:
Show that if $a < b$, then the closed interval $[a, b] \subset R$ does not have content zero.

Solution:
Since $[a, b]$ is a closed rectangle of volume $b - a$, then any covering of it by closed rectangles must have a total volume of at least $b-a$. Hence by setting $\varepsilon = b-a$ there will never be a cover (finite or countably infinite) of $[a, b]$ by closed rectangles with total volume less than this ε. Hence $[a, b]$ must not be a set of content zero.

6 Problem 9 from Slides

Problem:
Show that a compact set $A \subset R^n$ has measure zero if and only if it has content zero.

Solution:
Let A be a compact set with measure zero. Let $\varepsilon > 0$, then in light of problem 4 there is an uncountable collection of open rectangles $\mathscr{U} = \{U_i\}$ with $\mathrm{v}(\mathscr{U}) < \varepsilon$. However, since A is compact there is a finite subcollection of $\mathscr{U}$, say $U_{n_1}, U_{n_2}, \ldots, U_{n_k}$, which covers A. Furthermore, this subcollection has the property

$$\mathrm{v}\left(U_{n_1}\right) + \mathrm{v}\left(U_{n_2}\right) + \cdots + \mathrm{v}\left(U_{n_k}\right) \leq \mathrm{v}(\mathscr{U}) < \varepsilon$$

which implies that A has content zero.

The converse is trivial as content zero implies measure zero.

7 Problem 10 from Slides

Problem:
Show that the set A of rational numbers between 0 and 1 does not have content zero.

Solution:
Let $\mathscr{V} = \{V_i\}$ be a finite collection of n closed rectangles with total volume less than $1/2$. Denote each V_i by $[a_i, b_i]$ and define $j = \operatorname{argmin}_i\{a_i\}$. Note we know j exists due to the finite cardinality of $\mathscr{V}$. We then have two possible scenarios.

1. $a_j > 0$: If this is the case, then $(0, a_j)$ would be uncovered by $\mathscr{V}$. Since A is dense in $[0, 1]$ then there would be a point of A contained in $(0, a_j)$ and hence not be covered by $\mathscr{V}$.

2. $a_j \leq 0$: If this is the case, we may repeat our process developed here to determine if $A \cap (b_j, 1)$ is covered by $\mathscr{V} - V_j$. Given that $\mathscr{V}$ is finite, we will have two eventualities; either we will come across the previous case, or we will hit this current case for at most n times, ending when $\mathscr{V}$ is empty. If the former, we know $\mathscr{V}$ does not cover A, but if the latter, then $\mathscr{V}$ will cover $[0, x]$ and not (x, ∞) for some $x \geq 0$. However, given that $\mathrm{v}(\mathscr{V}) < 1/2$, then $x < 1/2$, i.e. $\mathscr{V}$ does not cover $(1/2, 1)$. Since A is dense in $[0, 1]$, then $A \cap (1/2, 1)$ is nonempty and, furthermore, not covered by $\mathscr{V}$.

Because all cases result in some subset of A remaining uncovered by $\mathscr{V}$ then there must be no finite set of closed rectangles that cover A and have volume less than $1/2$. Hence A does not have content zero.

8 Problem 11 from Slides

Problem:
Let $f : [a, b] \to R$ be an increasing function. Show that the set of all points $x \in [a, b]$ where f is discontinuous has measure zero.

Solution:
By Rudin's Theorem 4.30, the set of points E in $[a, b]$ where f is discontinuous is countable. So denote the points of E by $x_1, x_2, x_3, \ldots$. Thus for any $\varepsilon > 0$ we can cover E with closed rectangles $A_1, A_2, A_3, \ldots$ by

$$A_i = \left[x_i - \varepsilon\frac{1}{2^{i+1}}, x_i + \varepsilon\frac{1}{2^{i+1}}\right]$$

so that

$$\sum_i \mathrm{v}(A_i) = \sum_{i=1}^{\infty} \varepsilon\frac{1}{2^{i+1}} = \varepsilon\sum_{i=1}^{\infty}\frac{1}{2^{i+1}} = \frac{\varepsilon}{2} < \varepsilon$$

as desired.

9 Problem 12 from Slides

Problem:
Show that the bounded function $f : A \to R$ is continuous at $a \in A$ if and only if

$$\mathrm{o}(f, a) = \lim_{r\to 0}\left(M(a, f, r) - m(a, f, r)\right) = 0$$

Solution:
First assume that f is continuous. Let $\varepsilon > 0$. Then we can find a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \frac{\varepsilon}{2}$ for all $x \in A$. Hence if $r < \delta$ then

$$|M(a, f, r) - m(a, f, r)| \leq |M(a, f, r) - f(a)| + |f(a) - m(a, f, r)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

In other words $\lim_{r\to 0}\left(M(a, f, r) - m(a, f, r)\right) = 0$

Conversely, assume that $\lim_{r\to 0}\left(M(a, f, r) - m(a, f, r)\right) = 0$. Let $\varepsilon > 0$. Then we can find $\delta > 0$ such that $r < \delta$ implies that $|M(a, f, r) - m(a, f, r)| < \varepsilon$. Hence for any $x \in A$ with $|x - a| < \delta$ we have

$$|f(x) - f(a)| \leq |M(a, f, |x - a|) - m(a, f, |x - a|)| < \varepsilon$$

so that f is continuous.

10 Problem 13 from Slides

Problem:
Let $A \subset R^n$ be a closed set and $f : A \to R$ a bounded function. Show that the set $\{x \in A|\, \mathrm{o}(x, f) \geq \varepsilon\}$ is closed for any $\varepsilon > 0$.

Solution:
Let $\varepsilon > 0$ and put $B = \{x \in A|\, \mathrm{o}(x, f) \geq \varepsilon\}$. We show B is closed by showing it's complement is open. Let $b \in B^c$, in which case either $b \notin A$ or both $b \in A$ and $o(f, b) < \varepsilon$. If the former then since A is closed there's a neighborhood of b contained in A^c which is contained in B^c implying B is closed in this case. So assume the latter. Then $\lim_{r\to 0} (M(b, f, r) - m(b, f, r)) = \ell$ for some ℓ with $0 \leq \ell < \varepsilon$. Hence there exists a $\delta > 0$ such that $r < \delta$ implies $M(b, f, r) - m(b, f, r) - \ell < \epsilon - \ell$, i.e.

$$M(b, f, r) - m(b, f, r) < \varepsilon \tag{10.11}$$

whenever $r < \delta$. Let $y \in B_{\delta/2}(b)$ where $B_{\delta/2}(b)$ is the open ball around b of radius $\delta/2$. Then we have $B_{\delta/4}(y) \subset B_{\delta/2}(b)$ which together with equation 10.11 implies

$$M(y, f, r) - m(y, f, r) \leq M\left(b, f, {}^{\delta}\!/_{2}\right) - m\left(b, f, {}^{\delta}\!/_{2}\right) < \varepsilon$$

whenever $r < \delta/4$. In other words,

$$\lim_{r\to 0} (M(y, f, r) - m(y, f, r)) < \varepsilon$$

Hence $y \in B^c$. Since $y \in B_{\delta/2}(b)$ was arbitrary, then $B_{\delta/2}(b) \subset B^c$, implying that B^c is open and it's complement B is closed, as desired.

11 Problem 14 from Slides

Problem:
Let $A \subset R^n$ be a closed rectangle and $f : A \to R$ a bounded function such that for all $x \in A$, $\mathrm{o}(f, x) < \varepsilon$ for a fixed $\varepsilon > 0$. Show that there is a partition P of A such that $U(f, P) - L(f, P) < \varepsilon\, \mathrm{v}(A)$.

Solution:
Since $\mathrm{o}(f, x) < \varepsilon$ for all $x \in A$, then putting

$$\ell_x = \lim_{r\to 0} (M(x, f, r) - m(x, f, r))$$

for each $x \in A$ yields $\ell_x < \varepsilon$, i.e. $\varepsilon - \ell_x > 0$ for each $x \in A$. The above equation then implies that for each $x \in A$ there is a $\delta_x > 0$ such that $M(x, f, r) - m(x, f, r) - \ell_x < \varepsilon - \ell_x$ whenever $r < \delta_x$, in other words $M(x, f, r) - m(x, f, r) < \varepsilon$ whenever $r < \delta_x$. Thus by setting $\delta = \inf_x\{\delta_x\}$ we have

$$M(x, f, r) - m(x, f, r) < \varepsilon \tag{11.12}$$

for all $x \in A$ whenever $r < \delta$. For later ease of notation, put $\eta = \delta/2$, *noting that therefore $\eta < \delta$ and so equation 11.12 applies for $r = \eta$.*

Now denote A by $[a_1, b_1] \times \cdots \times [a_n, b_n]$ and define a partition $P = (P_1, \ldots, P_n)$ of A by setting

$$P_i = \{a_i, a_i + k_i, a_i + 2k_i, \ldots, a_i + (n-1)k_i, b_i\}$$

where we define $k_i = \frac{b_i - a_i}{m}$, and m is chosen so that $b_i - a_i < m\left({}^{\eta}\!/_{\sqrt{2}}\right)$ for all i. Defining P in this way ensures that each rectangle $S \in P$ has sides of length less than $\frac{\eta}{\sqrt{2}}$. This implies that for each such rectangle there's an $x_S \in A$

with $S \subset B_\eta(x_S)$ where $B_\eta(x_S)$ is the open ball of radius η centered at x_S. Hence equation 11.12 implies

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{S \in P} \left(M_S(f) - m_S(f)\right) \mathrm{v}(S) \\ &\leq \sum_{S \in P} \left(M(x_S,f,\eta) - m(x_S,f,\eta)\right) \mathrm{v}(S) \\ &< \sum_{S \in P} \varepsilon \, \mathrm{v}(S) \\ &= \varepsilon \sum_{S \in P} \mathrm{v}(S) \\ &= \varepsilon \, \mathrm{v}(A) \end{aligned}$$

as desired.

12 Problem 20 from Slides

Define $f : R \to R$ by

$$f(x) = \begin{cases} e^{-x^{-2}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and define $g : R \to R$ by

$$g(x) = f(x-a)f(b-x)$$

for some real numbers $a < b$.

(a) Prove that f is of class C^∞

We will prove that all orders of derivatives of f have the form $p(x)f(x)$ where $p(x)$ is a polynomial. Doing this shows that f is of class C^∞ since the product of a polynomial and f is both differentiable and continuous. We first see that as a base case $f(x) = e^{-x^{-2}}$ is already of the form $p(x)f(x)$ for $p(x) = 1$. So now let

$$f^{(n)}(x) = p(x)f(x) \tag{12.13}$$

for some polynomial $p(x)$. Since $f'(x) = 2x^{-3}f(x)$, then

$$f^{(n+1)}(x) = p'(x)f(x) + p(x)f'(x) = p'(x)f(x) + 2x^{-3}p(x)f(x) = \left(p'(x) + 2x^{-3}p(x)\right) f(x)$$

so that $f^{(n+1)}(x)$ is the product of f and a polynomial. The inductive hypothesis thus tells us that 12.13 holds for all positive n, as desired.

(b) Prove that g is of class C^∞ and positive on (a, b) but zero elsewhere

We will prove that all orders of derivatives of g have the form $p(x)g(x)$ where $p(x)$ is a polynomial. Doing this shows that g is of class C^∞ since the product of a polynomial and $g(x)$ is both differentiable and continuous. As a base case we have that $g(x)$ is already of the form $p(x)g(x)$ for $p(x) = 1$. So now let

$$g^{(n)}(x) = p(x)g(x) \tag{12.14}$$

for some polynomial $p(x)$. Since $g'(x) = (2(x-a)^{-3} - 2(b-x)^{-3})g(x)$, then

$$\begin{aligned} g^{(n+1)}(x) &= p'(x)g(x) + p(x)g'(x) \\ &= p'(x)g(x) + p(x)(2(x-a)^{-3} - 2(b-x)^{-3})g(x) \\ &= \left(p'(x) + (2(x-a)^{-3} - 2(b-x)^{-3})p(x)\right) g(x) \end{aligned}$$

so that $g^{(n+1)}(x)$ is the product of g and a polynomial. The inductive hypothesis thus tells us that 12.14 holds for all positive n, as desired.

Furthermore, for any $x_0 \leq a$ we have $x_0 - a \leq 0$ so that $f(x_0 - a) = 0$ which in turn means $g(x_0) = 0$. Likewise, when $x_0 \geq b$ then $b - x_0 \leq 0$ so that $f(b - x_0) = 0$ implying $g(x_0) = 0$. Finally, whenever $x_0 \in (a, b)$ we have both $0 < b - x_0$ and $0 < x_0 - a$ implying that $g(x_0) = f(x_0 - a)f(b - x_0) = e^{-(x_0-a)^{-2} - (b-x_0)^{-2}} > 0$ so that $g(x_0)$ is positive in this case.

(c)

Put

$$M = \int_{-\infty}^{\infty} g(x)dx$$

and then define $h : R \to R$ by

$$h(x) = \frac{1}{M} \int_{-\infty}^{x} g(x)dx$$

Show h is of class C^∞

Since $g \in C^\infty$, then the fact that

$$h'(x) = \frac{1}{M} g(x)$$

implies that h is of class C^∞.

Show $h(x) = 0$ for $x \leq a$

According to the previous part of this problem, $g(x) = 0$ for $x \leq a$, so that for some $c \in R$

$$h(x) = \frac{1}{M} \int_{-\infty}^{x} 0dx = \frac{1}{M} 0\Big|_{-\infty}^{x} = \frac{1}{M}(c - c) = 0$$

whenever $x \leq a$.

Show $0 < h(x) < 1$ for $a < x < b$

According to the previous part of this problem, $g(x) > 0$ for all $x \in (a, b)$ and zero elsewhere. Hence

$$\int_{-\infty}^{x} g(x)dx > 0 \tag{12.15}$$

and

$$\int_{x}^{\infty} g(x)dx > 0 \tag{12.16}$$

Inequality 12.15 allows us to add the value on its left to both sides of inequality 12.16. Doing so yields our desired upper bound.

$$\begin{aligned}
\int_{-\infty}^{x} g(x)dx + \int_{x}^{\infty} g(x)dx &> \int_{-\infty}^{x} g(x)dx \\
\int_{-\infty}^{\infty} g(x)dx &> \int_{-\infty}^{x} g(x)dx \\
M &> \int_{-\infty}^{x} g(x)dx \\
1 &> \frac{1}{M}\int_{-\infty}^{x} g(x)dx \\
1 &> h(x)
\end{aligned}$$

Furthermore, adding together 12.15 and 12.16 yields

$$\int_{-\infty}^{x} g(x)dx + \int_{x}^{\infty} g(x)dx = \int_{-\infty}^{\infty} g(x)dx > 0$$

so that $M > 0$. Combining this with inequality 12.15 gives us the lower bound we desire.

$$\begin{aligned}
\frac{1}{M}\int_{-\infty}^{x} g(x)dx &> \frac{1}{M}(0) \\
\frac{1}{M}\int_{-\infty}^{x} g(x)dx &> 0 \\
h(x) &> 0
\end{aligned}$$

Show $h(x) = 1$ for $x \geq b$

Since $g(x) = 0$ whenever $x \geq b$, then for $x \geq b$ we have

$$\begin{aligned}
\int_{-\infty}^{x} g(x)dx &= \int_{-\infty}^{\infty} g(x)dx \\
\frac{\int_{-\infty}^{x} g(x)dx}{\int_{-\infty}^{\infty} g(x)dx} &= 1 \\
\frac{1}{M}\int_{-\infty}^{x} g(x)dx &= 1 \\
h(x) &= 1
\end{aligned}$$

(d)

Let $a, b \in R$ be such that $a < b$. Define a function $k : R^n \to R$ by

$$k(\mathbf{x}) = 1 - h(|\mathbf{x}|)$$

Then given the properties of h we proved above, we obtain the following properties of k:

- When $|\mathbf{x}| \leq a$ $h(|\mathbf{x}|) = 0$ so that $k(\mathbf{x}) = 1 - 0 = 1$
- When $a < |\mathbf{x}| b$ $0 < h(|\mathbf{x}|) < 1$ so that

$$\begin{aligned} 0 >& - h(\mathbf{x}) > -1 \\ 1 >& 1 - h(\mathbf{x}) > 0 \\ 1 >& k(\mathbf{x}) > 0 \end{aligned}$$

- When $|\mathbf{x}| \geq b$ $h(|\mathbf{x}|) = 1$ so that $k(\mathbf{x}) = 1 - 1 = 0$

Furthermore, since the partial derivative of k with respect to x_i is

$$\frac{\partial}{\partial x_i} k = \frac{x_i}{|\mathbf{x}|} h'(|\mathbf{x}|)$$

then the fact that h is of class C^∞ implies that k is also of the class C^∞, as desired.