Math 509: Advanced Analysis
Homework 7

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1 Fully prove steps 1 through 4 of the Change of Variables Theorem

Change of Variables Theorem: Let $A \subset R^n$ be an open set and $g : A \to R^n$ a one-to-one, continuously differentiable map such that $\det g'(x) \neq 0$ for all $x \in A$. If $f : g(A) \to R$ is a Riemann integrable function, then

$$\int_{g(A)} f = \int_A (f \circ g)|\det g'|$$

Proof: The proof begins with several reductions which allow us to assume that $f \equiv 1$, that A is a small open set about a point a, and that $g'(a)$ is the identity matrix. Then the argument is completed by induction on n with the use of Fubini's Theorem.

(a) Step 1

Suppose there is an open cover $\mathcal{V}$ of A such that for each $U \in \mathcal{V}$ and any integrable f, we have

$$\int_{g(U)} f = \int_U (f \circ g)|\det g'|$$

Then the theorem is true for all A.

Proof.
The collection of all $g(U)$ is an open cover of $g(A)$. Let Φ be a partition of unity subordinate to this cover. For any Riemann integrable $f : g(A) \to R$, if $\varphi = 0$ outside of $g(U)$, then, since g is one-to-one we have that $(\varphi f) \circ g = 0$ outside of U. Hence φf is integrable and the equation

$$\int_{g(U)} \varphi f = \int_U ((\varphi f) \circ g)|\det g'|$$

can be written as

$$\int_{g(A)} \varphi f = \int_A ((\varphi f) \circ g)|\det g'|$$

Summing over all $\varphi \in \Phi$ yields

$$\sum_{\varphi \in \Phi} \int_{g(A)} \varphi f = \sum_{\varphi \in \Phi} \int_A ((\varphi f) \circ g)|\det g'|$$

$$\int_{g(A)} \left(\sum_{\varphi \in \Phi} \varphi \right) f = \int_A \left(\sum_{\varphi \in \Phi} ((\varphi f) \circ g) \right) |\det g'|$$

$$\int_{g(A)} f = \int_A \left(\left(\left(\left(\sum_{\varphi \in \Phi} \varphi \right) f \right) \circ g \right) \right) |\det g'|$$

$$\int_{g(A)} f = \int_A (f \circ g)|\det g'|$$

as desired.

(b) Step 2

It suffices to prove the theorem for $f = 1$.

Proof.
If the theorem holds for $f = 1$ then it holds for $f =$ constant. Let V be a rectangle in $g(A)$ and P a partition of V.

For each subrectangle S of P, let f_S be the constant function $m_S(f)$. Then

$$\begin{aligned}
L(f,P) &= \sum_{S\in P} m_S(f)v(S) \\
&= \sum_{S\in P} \int_{\text{int}S} f_S \\
&= \sum_{S\in P} \int_{g-1(\text{int}S)} (f_S \circ g)|\det g'| \\
&\leq \sum_{S\in P} \int_{g-1(\text{int}S)} (f \circ g)|\det g'| \\
&= \int_{g-1(V)} (f\circ g)|\det g'|
\end{aligned}$$

Since $\int_V f = \text{LUB}_P(f,P)$, this proves that

$$\int_V f \leq \int_{g-1(V)} (f\circ g)|\det g'|$$

Likewise, letting $f_S = M_S(f)$, we get the opposite inequality, and so that conclude that

$$\int_V f = \int_{g-1(V)} (f\circ g)|\det g'|$$

Then as in Step 1, it follows that

$$\int_{g(A)} f = \int_A (f\circ g)|\det g'|$$

(c) Step 3

If the theorem is true for $g : A \to R^n$ and for $h : B \to R^n$ where $g(A) \subset B$, then it is also true for $h \circ g : A \to R^n$.

Proof.
To ease the proof slightly, define $X = g(A)$ and $f' = (f\circ h)|\det h'|$. Then we have

$$\begin{aligned}
\int_{h\circ g(A)} f &= \int_{h(g(A))} f \\
&= \int_{h(X)} f \\
&= \int_X (f\circ h)|\det h'| \\
&= \int_X f' \\
&= \int_{g(A)} f' \\
&= \int_A (f'\circ g)|\det g'| \\
&= \int_A \left(((f\circ h)|\det h'|)\circ g\right)|\det g'| \\
&= \int_A ((f\circ h)\circ g)\left(|\det h'|\circ g\right)|\det g'| \\
&= \int_A (f\circ(h\circ g))\left(|\det h'|\circ g\right)|\det g'| \\
&= \int_A (f\circ(h\circ g))|\det(h\circ g)'|
\end{aligned}$$

as desired.

(d) Step 4

The theorem is true if g is a linear tranformation.

Proof.
By steps 1 and 2, it suffices to show for any open rectangle U that

$$\int_{g(U)} 1 = \int_U |\det g'|$$

Note that for a linear transformation g, we have $g' = g$. Then this is just the fact from linear algebra that a linear tranformation $g : R^n \to R^n$ multiplies volumes by $|\det g|$.

2 Fully prove the Fundamental Theorem

Let A be a closed rectangle in R^n and $f : A \to R$ a bounded function. Let

$$B = \{x \in A : f \text{ is not continuous at } x\}$$

Then f is Riemann integrable on A if and only if B has measure zero.

Proof.
Suppose first that B has measure zero.

Let $\varepsilon > 0$. Define $B_\varepsilon = \{x \in A : o(f, x) \geq \varepsilon\}$. Now $B_\varepsilon \subset B$, hence B_ε has measure zero. By problem 13 of our previous Chapter 2, the set B_ε is closed. Since B_ε is also bounded, it is compact, and so has content zero. Hence there is a finite collection $U_1, \ldots, U_n$ of closed rectangles, whose interiors cover B_ε, with total volume less than ε.

Now let P be a partition of the original rectangle A which "refines" the collection of rectangles U_i in the following sense. Each rectangle $S \in P$ is in one of the following two groups:

1. Group 1 (G_1): $S \subset U_i$ for some i
2. Group 2 (G_2): otherwise; i.e. S is disjoint from B_ε

Since the function f is, by hypothesis, bounded on A, choose M so that $|f(x)| < M$ for all $x \in A$. Then

$$M_S(f) - m_S(f) < 2M$$

for all $S \in P$. Thus since

$$U(f, P) - L(f, P) = \sum_{S \in P} [M_S(f) - m_S(f)] \operatorname{vol}(S)$$

we can divide the above difference into two parts, the first corresponding to G_1 and the other to G_2. We have the following for the first part

$$\sum_{S \in G_1} [M_S(f) - m_S(f)] \operatorname{vol}(S) < 2M \sum_i \operatorname{vol}(U_i) < 2M\varepsilon \tag{2.1}$$

As for the second part, since each point $x \in S \in G_2$ has $o(f, x) < \varepsilon$, then any $S \in G_2$ can be further partitioned into rectangles S' so that

$$\sum_{S' \subset S} [M_{S'}(f) - m_{S'}(f)] \operatorname{vol}(S') < \sum_{S' \subset S} \varepsilon \operatorname{vol}(S') < \varepsilon \sum_{S' \subset S} \operatorname{vol}(S') < \varepsilon \operatorname{vol}(S)$$

Thus replacing the partitions in G_2 with these refined partitions implies the following bound

$$\sum_{S' \in G_2} [M'_S(f) - m'_S(f)] \operatorname{vol}(S') < \sum_{S \in G_2} \varepsilon \operatorname{vol}(S) = \varepsilon \sum_{S \in G_2} \operatorname{vol}(S) < \varepsilon \operatorname{vol}(A) \tag{2.2}$$

Putting together the partial sums from G_1 and G_2 of equations 2.1 and 2.2 yields

$$U(f, P) - L(f, P) = \sum_{S \in P} [M_S(f) - m_S(f)] \operatorname{vol}(S) < 2M\varepsilon + \varepsilon \operatorname{vol}(A)$$

The value on the right-hand side can be made arbitrarily small by appropriate choice of ε and so we conclude that f is Riemann integrable.

Conversely, suppose that f is Riemann integrable. We must show that the set B has measure zero. Since $B = B_1 \cup B_{1/2} \cup B_{1/3} \cup \cdots$, it is enough to show that each $B_{1/n}$ has measure zero.

Since $B_{1/n}$ is compact, that is the same as having content zero. Since f is Riemann integrable, then given any $\varepsilon > 0$ we can find a partition P of A such that

$$U(f, P) - L(f, P) < \frac{\varepsilon}{n}$$

Let G be the subfamily of P consisting of rectangles which meet $B_{1/n}$. Then the rectangles S in G cover $B_{1/n}$. Expand slightly each of these rectangles S to a rectangles S', so that the interiors of the S' now cover $B_{1/n}$. Then each of the rectangles S' contains in its interior a point $x \in B_{1/n}$ where the oscillation $o(f, x) \geq 1/n$. It follows from this that

$$M_{S'}(f) - m_{S'}(f) \geq 1/n$$

Hence

$$(1/n) \sum_{S' \in G'} \mathrm{vol}(S') \leq \sum_{S' \in G'} [M_{S'}(f) - m_{S'}(f)] \, \mathrm{vol}(S') \leq \sum_{S' \in P'} [M_{S'}(f) - m_{S'}(f)] \, \mathrm{vol}(S') < \varepsilon/n$$

and therefore $\sum_{S' \in G'} \mathrm{vol}(S') < \varepsilon$. Since the rectangles S' in G' cover $B_{1/n}$, and since $\varepsilon > 0$ is arbitrarily small, this shows the $B_{1/n}$ has content zero, completing the proof.