Math 509: Advanced Analysis
Homework 11

Lawrence Tyler Rush
<me@tylerlogic.com>

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1 Multilinear Algebra Problem 2

Problem:
Let $v_1, \ldots, v_n$ be a basis for V and $\varphi_1, \ldots, \varphi_n$ be the dual basis for $V^* = \mathcal{T}^1(V)$. Show that the set

$$\mathcal{A} = \{\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} | 1 \leq i_1, \ldots, i_k \leq n\}$$

is a basis for the set of all k-fold tensor products, $\mathcal{T}^k(V)$,

Solution:
We first state and prove a useful lemma.

Lemma 1. *For any m-dimensional vector space W with basis $\psi_1, \ldots, \psi_m$, the set*

$$\mathcal{B} = \{\psi_i \otimes \varphi_j \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

is a basis for $W \otimes V^*$.

Proof. Let $\alpha \otimes \beta \in W \otimes V^*$. Then there are reals $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ such that $\alpha = a_1\psi_1 + \cdots + a_m\psi_m$ and $\beta = b_1\varphi_1 + \cdots + b_n\varphi_n$. Hence

$$\begin{aligned}
\alpha \otimes \beta &= \left(\sum_{i=1}^{m} a_i\psi_i\right) \otimes \beta \\
&= \sum_{i=1}^{m} (a_i\psi_i \otimes \beta) \\
&= \sum_{i=1}^{m} a_i (\psi_i \otimes \beta) \\
&= \sum_{i=1}^{m} a_i \left(\psi_i \otimes \left(\sum_{j=1}^{n} b_j\varphi_j\right)\right) \\
&= \sum_{i=1}^{m} a_i \left(\sum_{j=1}^{n} (\psi_i \otimes b_j\varphi_j)\right) \\
&= \sum_{i=1}^{m} a_i \left(\sum_{j=1}^{n} b_j (\psi_i \otimes \varphi_j)\right) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j (\psi_i \otimes \varphi_j)
\end{aligned}$$

implying that $\mathcal{B}$ spans $W \otimes V^*$. Now if there existed reals c_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq n$ such that

$$\alpha \otimes \beta = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} (\psi_i \otimes \varphi_j)$$

then we'd have

$$\begin{aligned}
0 &= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j (\psi_i \otimes \varphi_j) - \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} (\psi_i \otimes \varphi_j) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} (a_i b_j - c_{ij}) (\psi_i \otimes \varphi_j)
\end{aligned}$$

Since each $\psi_i \otimes \varphi_j$ is nonzero, we must have that each $a_i b_j - c_{ij} = 0$, implying that $\mathcal{B}$ is linearly independent. Hence $\mathcal{B}$ is a basis. Hence $\mathcal{B}$ is a basis. □

We can now prove $\mathcal{A}$ is a basis for $\mathcal{T}^k(V)$ by induction. As a base case, we have that $\{\varphi_i\}$ is a basis for $T^1(V)$ since $T^1(V) = V^*$. Assuming, now, that $\{\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_{k-1}}\}$ is a basis for $T^{k-1}(V)$, then lemma 1 informs us that the set

$$\left\{\psi \otimes \varphi_j \mid \psi \in \{\varphi_{i_1} \otimes \varphi_{i_{k-1}}\},\ 1 \leq j \leq n\right\}$$

is a basis for $T^{k-1}(V) \otimes V^* = T^k(V)$. But this set is just $\mathcal{A}$, and thus $\mathcal{A}$ must be a basis for $T^{k-1}(V) \otimes V^* = T^k(V)$, as desired.

2 Multilinear Algebra Problem 3

For a k-tensor $T \in \mathcal{T}^k(V)$, define

$$\operatorname{Alt}(T)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma T(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

for any $v_1, \ldots, v_k \in V$.

(a) Show that $\operatorname{Alt}(T)$ is alternating.

Let i, j be integers with $1 \leq i < j \leq k$. For each $\sigma \in S_k$, define $\tau_\sigma = \sigma \circ (i\ j)$ where (ij) is the element of S_k that transpositions i and j. Hence we have

$$(-1)^{\tau_\sigma} = -(-1)^\sigma$$

Furthermore, we have

$$\tau_\sigma(n) = \sigma \circ (i\ j)(n) = \begin{cases} \sigma(i) & n = j \\ \sigma(j) & n = i \\ \sigma(n) & \text{otherwise} \end{cases}$$

for any integer $n \in \{1, \ldots, k\}$. Making use of these two equations and the fact that the action of $(i\ j)$ on S_k simply permutes the elements of S_k, we see that for $v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k \in V$ we have

$$\begin{aligned} \operatorname{Alt}(T)(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k) &= \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma T(v_{\sigma(1)}, \ldots, v_{\sigma(j)}, \ldots, v_{\sigma(i)}, \ldots, v_{\sigma(k)}) \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} -(-1)^{\tau_\sigma} T(v_{\tau_\sigma(1)}, \ldots, v_{\tau_\sigma(i)}, \ldots, v_{\tau_\sigma(j)}, \ldots, v_{\tau_\sigma(k)}) \\ &= -\frac{1}{k!} \sum_{\tau_\sigma \in S_k} (-1)^{\tau_\sigma} T(v_{\tau_\sigma(1)}, \ldots, v_{\tau_\sigma(i)}, \ldots, v_{\tau_\sigma(j)}, \ldots, v_{\tau_\sigma(k)}) \\ &= -\operatorname{Alt}(T)(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) \end{aligned}$$

as desired.

(b) Show that $\operatorname{Alt}(T) = T$ whenever T is alternating

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Denote the set of even and odd permutations of S_k by S_k^+ and S_k^-, respectively. Hence

$$\begin{aligned} T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) &= T(v_1, \ldots, v_k) && \forall \sigma \in S_k^+ \\ T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) &= -T(v_1, \ldots, v_k) && \forall \sigma \in S_k^- \end{aligned}$$

In particular, we note S_k^+ and S_k^- have the same cardinality, $k!/2$, and that they partition S_k so that

$$\begin{aligned}
\mathrm{Alt}(T)(v_1,\dots,v_k) &= \frac{1}{k!}\sum_{\sigma\in S_k}(-1)^\sigma T(v_{\sigma(1)},\dots,v_{\sigma(k)}) \\
&= \frac{1}{k!}\left(\sum_{\sigma\in S_k^+}(-1)^\sigma T(v_{\sigma(1)},\dots,v_{\sigma(k)}) + \sum_{\sigma\in S_k^-}(-1)^\sigma T(v_{\sigma(1)},\dots,v_{\sigma(k)})\right) \\
&= \frac{1}{k!}\left(\sum_{\sigma\in S_k^+} T(v_{\sigma(1)},\dots,v_{\sigma(k)}) - \sum_{\sigma\in S_k^-} T(v_{\sigma(1)},\dots,v_{\sigma(k)})\right) \\
&= \frac{1}{k!}\left(\sum_{\sigma\in S_k^+} T(v_1,\dots,v_k) - \sum_{\sigma\in S_k^-} -T(v_1,\dots,v_k)\right) \\
&= \frac{1}{k!}\left(\sum_{\sigma\in S_k^+} T(v_1,\dots,v_k) + \sum_{\sigma\in S_k^-} T(v_1,\dots,v_k)\right) \\
&= T(v_1,\dots,v_k)\frac{1}{k!}\left(\sum_{\sigma\in S_k^+} 1 + \sum_{\sigma\in S_k^-} 1\right) \\
&= T(v_1,\dots,v_k)\frac{1}{k!}\left(k!/2 + k!/2\right) \\
&= T(v_1,\dots,v_k)
\end{aligned}$$

as desired.

(c) Conclude that $\mathrm{Alt}(\mathrm{Alt}(T)) = \mathrm{Alt}(T)$

Since $\mathrm{Alt}(T)$ is alternating for any T, then the previous part of this problem yields $\mathrm{Alt}(\mathrm{Alt}(T)) = \mathrm{Alt}(T)$

3 Multilinear Algebra Problem 7

4 Differential Forms Problem 1

Fix an arbitrary point $p \in R^n$. Then $d\mathbf{x}^1(p),\dots,d\mathbf{x}^n(p)$ is a basis for the vector space $\mathrm{Hom}(R^n_p, R)$, so that

$$df(p) = f_1(p)d\mathbf{x}^1(p) + \cdots + f_n(p)d\mathbf{x}^n(p)$$

where each f_i is a real valued function on R^n. In other words

$$df = f_1 d\mathbf{x}^1 + \cdots + f_n d\mathbf{x}^n$$

so we need only find these functions f_i. However, because $d\mathbf{x}^1(p),\dots,d\mathbf{x}^n(p)$ is the dual basis to the basis $(e_1)_p,\dots,(e_n)_p \in R^n_p$, then we know $f_i(p) = df(p)(e_i)_p$ for each i. Thus

$$\begin{aligned}
f_i(p) &= df(p)(e_i)_p \\
&= f'(p)(e_i) \\
&= \left(\frac{\partial f}{\partial x^1}(p)\ \frac{\partial f}{\partial x^2}(p)\ \cdots\ \frac{\partial f}{\partial x^n}(p)\right)(e_i) \\
&= \frac{\partial f}{\partial x^i}(p)
\end{aligned}$$

Hence $f_i = \frac{\partial f}{\partial x^i}$ indicating

$$df = \left(\frac{\partial f}{\partial x^i}\right) d\mathbf{x}^1 + \cdots + \left(\frac{\partial f}{\partial x^i}\right) d\mathbf{x}^n$$

as desired.

5 Differential Forms Problem 2

Let $f : R^m \to R^n$ be a differentiable function.

(a) Show that $f^*(dy^i) = \sum_{j=1}^n \left(\frac{\partial f^i}{\partial x^j}\right) dx^j$

For fixed $p \in R^m$ we have

$$\begin{aligned}(f^*(dy^i))(p) &= dy^i(f(p)) \\ &= \sum_{j=1}^n \left(\frac{\partial f^i}{\partial x^j}\right) dx^j(p)\end{aligned}$$

so that

$$f^*(dy^i) = \sum_{j=1}^n \left(\frac{\partial f^i}{\partial x^j}\right) dx^j$$

as desired.

(b) Show that $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$

By the definition of $f^*\omega$ and the linearity of f^*, we have

$$\begin{aligned}\left(f^*(\omega_1 + \omega_2)\right)(p) &= f^*\left((\omega_1 + \omega_2)(p)\right) \\ &= f^*(\omega_1(p) + \omega_2(p)) \\ &= f^*(\omega_1(p)) + f^*(\omega_2(p)) \\ &= (f^*\omega_1)(p) + (f^*\omega_2)(p) \\ &= \left(f^*\omega_1 + f^*\omega_2\right)(p)\end{aligned}$$

for a fixed p.

(c) Show that $f^*(g\omega) = (g \circ f)f^*(\omega) = f^*(\omega)g^*(\omega)$

(d) Show that $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$

6 Differential Forms Problem 3

Let $f : R^2 \to R^2$ be defined by

$$f(u, v) = (x(u, v), y(u, v)) = (u^2 - v^2, 2uv)$$

and let $\omega = -ydx + xdy$. Thus we have

$$\begin{aligned}dx &= 2udu - 2vdv \\ dy &= 2vdu + 2udv\end{aligned}$$

so that

$$
\begin{aligned}
f^*\omega &= -(2uv)(2udu - 2vdv) + (u^2 - v^2)(2vdu + 2udv) \\
&= -4u^2vdu + 4uv^2dv + 2u^2vdu + 2u^3dv - 2v^3du - 2uv^2dv \\
&= (-2u^2v - 2v^3)du + (2uv^2 + 2u^3)dv
\end{aligned}
$$

7 Differential Forms Problem 4

Let $f : R^n \to R^n$ be a differentiable function. We endeavor to prove $f^*(dy^1 \wedge \cdots \wedge dy^n) = (\det f')dx^1 \wedge \cdots \wedge dx^n$ via induction. By problem 5(a) we have

$$
f^*(dy) = \left(\frac{\partial y}{\partial x}\right) dx
$$

yielding our base case for $n = 1$. Now assume that $n > 1$ and that

$$
f^*(dy^1 \wedge \cdots \wedge dy^k) = (\det f')dx^1 \wedge \cdots \wedge dx^k \tag{7.1}
$$

for any $k < n$. Letting J_n be the Jacobian of f without the n^{th} row and column, we therefore we have

$$
\begin{aligned}
f^*(dy^1 \wedge \cdots \wedge dy^{n-1} \wedge dy^n) &= f^*(dy^1 \wedge \cdots \wedge dy^{n-1}) \wedge f^*(dy^n) \\
&= \big((\det J_n)dx^1 \wedge \cdots \wedge dx^{n-1}\big) \wedge \left(\sum_{i=1}^{n} \left(\frac{\partial y^n}{\partial x^i}\right) dx^i\right) \\
&= \sum_{i=1}^{n} (\det J_n) \left(\frac{\partial y^n}{\partial x^i}\right) \left(dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dx^i\right)
\end{aligned}
$$

However, since alternating tensors are zero whenever there is a repeated component, the last line becomes

$$
f^*(dy^1 \wedge \cdots \wedge dy^{n-1} \wedge dy^n) = (\det J_n) \left(\frac{\partial y^n}{\partial x^n}\right) \left(dx^1 \wedge \cdots \wedge dx^n\right) = (\det f') \left(dx^1 \wedge \cdots \wedge dx^n\right)
$$

giving us the desired result.

8 Differential Forms Problem 5

9 Differential Forms Problem 6

(a)

(i)

Let $\omega = xdy + ydx$. Then the exterior derivative is

$$
\begin{aligned}
d\omega &= \left(\left(\frac{\partial x}{\partial x}\right) dx \wedge dy + \left(\frac{\partial x}{\partial y}\right) dy \wedge dy\right) + \left(\left(\frac{\partial y}{\partial x}\right) dx \wedge dx + \left(\frac{\partial y}{\partial y}\right) dy \wedge dx\right) \\
&= (dx \wedge dy + 0) + (0 + dy \wedge dx) \\
&= dx \wedge dy + dy \wedge dx \\
&= dx \wedge dy - dx \wedge dy \\
&= 0
\end{aligned}
$$

(ii)

Let $\omega = xdy - ydx$. Then the exterior derivative is

$$\begin{aligned}
d\omega &= \left(\left(\frac{\partial x}{\partial x}\right) dx \wedge dy + \left(\frac{\partial x}{\partial y}\right) dy \wedge dy\right) + \left(\left(\frac{\partial - y}{\partial x}\right) dx \wedge dx + \left(\frac{\partial - y}{\partial y}\right) dy \wedge dx\right) \\
&= (dx \wedge dy + 0) + (0 - dy \wedge dx) \\
&= dx \wedge dy - dy \wedge dx \\
&= dx \wedge dy + dx \wedge dy \\
&= 2(dx \wedge dy)
\end{aligned}$$

(b)

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10 Differential Forms Problem 8
